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ALGORITHMS FOR MINIMIZING TOTAL COST, BOTTLENECK TIME AND BOTTLENECK SHIPMENT IN TRANSPORTATION PROBLEMS

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ABSTRACT

We consider the transportation problem of determining nonnegative shipments from a set of m warehouses with given availabilities to a set of n markets with given requirements. Three objectives are defined for each solution: (i) total cost, TC, (ii) bottleneck time, BT (i.e., maximum transportation time for a positive shipment), and (iii) bottleneck shipment, SB (i.e., total shipment over routes with bottleneck time). An algorithm is given for determining all efficient (pareto-optimal or nondominated) (TC, BT) solution pairs. The special case of this algorithm when all the unit cost coefficients are zero is shown to be the same as the algorithms for minimizing BT provided by Szwarc and Hammer. This algorithm for minimizing BT is shown to be computationally superior. Transportation or assignment problems with $m=n=100$ average about a second on the UNIVAC 1108 computer (FORTRAN V)) to the threshold algorithm for minimizing BT. The algorithm is then extended to provide not only all the efficient (TC, BT) solution pairs but also, for each such BT, all the efficient (TC, SB) solution pairs. The algorithms are based on the cost operator theory of parametric programming for the transportation problem developed by the authors.

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1. INTRODUCTION

The transportation problem of linear programming [1, 3, 4] consists of finding nonnegative shipments $X=\{x_{ij}\}$ from the ith warehouse ($i\epsilon I=\{1, 2, \ldots, m\}$) to the jth market ($j\epsilon J=\{1, 2, \ldots, n\}$) satisfying the conditions:

(1) $$\sum_{j\epsilon J} x_{ij}=a_i \qquad \text{for } i\epsilon I$$

and

(2) $$\sum_{i\epsilon I} x_{ij}=b_j \qquad \text{for } j\epsilon J.$$

The warehouse capacities $\{a_i\}$ and market requirements $\{b_j\}$ are assumed to be nonnegative and such that

(3) $$\sum_{i\epsilon I} a_i=\sum_{j\epsilon J} b_j.$$

We define a *solution* X to be any set of nonnegative $\{x_{ij}\}$ satisfying (1) and (2). Denoting c_{ij} to be the unit cost and t_{ij} to be the time of transportation from warehouse i to market j (both c_{ij} and t_{ij} are assumed to be finite) we can define the objectives below for every solution X:

(4) Total Cost $$TC=\sum_{i\epsilon I}\sum_{j\epsilon J} c_{ij}x_{ij},$$

(5) Bottleneck Time $$BT=\max_{(i,j)|x_{ij}>0} t_{ij},$$

and

(6) Shipment along Bottleneck routes

$$SB=\sum_{(i,j)|t_{ij}=BT} x_{ij}.$$

Thus BT measures the maximum time taken by any shipping route *used* in the solution (i.e., $x_{ij}>0$), and SB gives the total shipment amount over all the routes with this maximum time. We will denote the problem of minimizing TC as the Standard Transportation Problem (STP) and that of minimizing BT as the Bottleneck Transportation Problem (BTP).

The STP has been one of the earliest and most fruitful applications of linear programming and several algorithms exist for solving it [1, 3, 4, 14]. Algorithms for the BTP have been provided by Szwarc [15, 16], Hammer [7, 8] and Garfinkel and Rao [5]. The primal algorithms (i.e., they maintain a (primal feasible) solution at every step) of Szwarc [15, 16] and Hammer [7] are essentially the same (see [16]) and will be shown to be a particular case of the more general algorithm provided in this paper. The Garfinkel-Rao approach is a "threshold" algorithm in that a solution is achieved only at termination of the procedure. They also discuss a primal algorithm, which they found to be computationally far inferior to the threshold algorithm. A thorough review of earlier East European literature on the BTP is provided in [16].

The problem of minimizing SB among all optimal solutions to the BTP can be solved by the algorithm of [7] after incorporating the corrections detailed in [8]. It can also be formulated as an STP by suitably defining the cost coefficients (see [5]).

Hammer [7] motivates the use of *BTP* in the context of military units dispatched from their a s to war fronts. The objective *BT* then gives the total time from the start of the operation until s ompletion. The same objective is also relevant in the delivery of emergency supplies to drought- r ood-affected areas. The same formulation is also meaningful in the transportation of perishable p modities. In each of these formulations, it is usually the case that among all optimal solutions) ie *BTP*, the solution that minimizes *SB* is preferred.

The *BTP* minimizes *BT* although this may lead to a large cost *TC*. Likewise the standard sportation problem minimizes *TC* although this may lead to a large value of *BT*. Similarly, ı. given value of *BT*, the objective of minimizing *TC* often conflicts with the objective of mini- ing *SB*.

The present paper advocates a more realistic approach of simultaneously considering more than objective. Since these objectives will, in general, be conflicting, we provide an approach for ining all *pareto-optimal* or *efficient* solution pairs (*TC*, *BT*). A solution pair (*TC*, *BT*) is efficient ere exists no other solution pair (*TC'*, *BT'*) such that either (i) $TC' < TC$ and $BT' \leq BT$ or $TC' \leq TC$ and $BT' < BT$. The present paper also provides an approach for obtaining all efficient ltion pairs (*TC*, *SB*) for a given value of *BT*. The decisionmaker can then subjectively choose "best" solution from among the efficient solutions. Our approach serves merely as an aid to sionmaking by eliminating all the inefficient solutions.

Although our discussion is given mainly in the transportation context, the problem formulation so useful in the problem of assigning n work stations located in an assembly line to n individuals. ning x_{ij} to be 1 or 0 depending on whether or not the ith station is assigned to the jth individual, constraints (1) and (2) with $a_i = b_j = 1$ state that each station has to be assigned to exactly one vidual and conversely. If we denote by t_{ij} the time in minutes (say) per assembly taken by the ndividual if he were to be assigned to the ith station, the objective *BT* (Equation (5)) gives the e time (i.e., the time interval in minutes between successive units coming out of the assembly so that the number of units of output from the assembly line is $1/BT$ per minute. Thus maxi- ng output is equivalent to minimizing *BT*. Now if we denote by p_{ij} the probability that the jth vidual will do a correct (i.e., not defective) assembly when he operates on the ith station, then $\sum_j p_{ij}x_{ij}$ gives the probability of a correct assembly at the ith station (recall that for any given i, $x_{ij} = 1$ for exactly one j). Consequently, assuming independence, the probability of a correct as- bly through the entire assembly line is $\theta = \prod_{i \in I}(\sum_{j \in J} p_{ij}x_{ij})$. Rather than maximizing this expres- we can equivalently (i.e., optimal solution $\{x_{ij}\}$ will be the same) maximize the logarithm of i., $ln\theta = \sum_{i \in I} ln(\sum_{j \in J} p_{ij}x_{ij})$. Defining $c_{ij} = -lnp_{ij} \geq 0$, the latter objective is equivalent to minimizing $= \sum_{i \in I}\sum_{j \in J} c_{ij}x_{ij}$ (this again follows from the fact that for any given i, $x_{ij} = 1$ for exactly one j). lly, if we assume that the assembly times have some random fluctuations (but with means) then the larger the value of *SB* (i.e., the number of men working at bottleneck time,) the er is the probability that the cycle time will exceed *BT*. Consequently in the assembly line xt, the three objectives, viz., minimizing *TC*, minimizing *BT* and minimizing *SB*, are all ant. The assignment formulation with objectives *TC* and *BT* (and to an extent *SB*) is also ant in assigning men to jobs based on the "fit" of job requirements to individual qualifications the discussion of the c and l_1 metrics in [2, 13]).

After the preliminary definitions in Section 2, Section 3 provides a procedure for obtaining a the efficient solution pairs (TC, BT). The algorithm of Section 3 specializes to the Szwarc-Hamme algorithm for the BTP [7, 15, 16] for the particular case when all $c_{ij}=0$. Section 4 provides com putational results for this algorithm for BTP and shows that its computational efficiency compare favorably with the threshold algorithm [5]. Finally, Section 5 provides a procedure for determinin all the efficient solution pairs (TC, BT) and for each such BT, all the efficient solution pairs (TC SB).

2. PRELIMINARY DEFINITIONS AND RESULTS

The transportation problem of determining nonnegative $\{x_{ij}\}$ satisfying the constraint (1)–(2) is usually represented in tabular form with the set I of warehouses represented by m *row* and the set J of markets represented by n *columns*. A *line* refers to a row or a column. By a *cell* w mean an index pair (i, j) with $i\epsilon I$ and $j\epsilon J$.

DEFINITION 1: Let Ω denote a collection of cells. *Line g is said to be connected to line h in* if and only if there exists a *path* (sequence) S of distinct cells belonging to Ω:

$$S=\{(i_1, j_1), (i_2, j_2), \ldots, (i_k, j_k)\} \tag{7}$$

such that

(a) (i_1, j_1) is the only cell of S in line g and (i_k, j_k) is the only cell of S in line h.

(b) for every $1<t<k$ either (i) or (ii) below holds:

(i) $i_t=i_{t-1}$ and $j_t=j_{t+1}$

(ii) $j_t=j_{t-1}$ and $i_t=i_{t+1}$.

REMARK 1: From the properties (a) and (b) of Definition 1, it follows that the number c cells in the path (7) connecting lines g and h is even if both g and h are rows (columns); it is odd if is a row and h is a column or vice versa. If g is a row, h is a column and $(g, h) \notin \Omega$, then $k\geq 3$.

We assume that the reader is familiar with the terminology of the simplex method for th standard transportation problem [1, 3], viz., *basis* B (set of $m+n-1$ cells such that every pair o lines is connected in B), *solution* X, *basic solution* (i.e., a solution X such that $(i, j) \notin B \Rightarrow x_{ij}=0$ We will define $\{u_i\}$ and $\{v_j\}$ to be the *dual variables* (corresponding to rows $i\epsilon I$ (Equation (1)) an columns $j\epsilon J$ (Equation (2)) respectively) for the basis B so that

$$c_{ij}=u_i+v_j \qquad \text{for } (i, j) \in B. \tag{8}$$

The basis is *dual feasible* if, and only if,

$$c_{ij}\geq u_i+v_j \qquad \text{for } i\epsilon I \text{ and } j\epsilon J. \tag{9}$$

A basic solution which is dual feasible is *optimal*. A *cycle* Γ is created when a nonbasic ce (e, f) is added to B. The cycle is $\Gamma=\{(e, f)\} \cup S$, where S is the unique path from row e to column in B. By Remark 1, Γ has an even number of cells (≥ 4)

$$\Gamma=\{(e, f)=(i_0, j_0)\} \cup \{(i_1, j_1), (i_2, j_2), \ldots, (i_{2l-1}, j_{2l-1})\}. \tag{10}$$

Let

$$\Gamma_1=\{(i_1, j_1), (i_3, j_3), \ldots, (i_{2l-1}, j_{2l-1})\} \tag{11}$$

and

$$\Gamma_2=\{(i_0, j_0), (i_2, j_2), \ldots, (i_{2l-2}, j_{2l-2})\}. \tag{12}$$

define Γ_1 as the cells with *odd parity* and Γ_2 as the cells with *even parity* each set having exactly (≥ 2) members.

REMARK 2: By Definition 1, it follows that every row $i \epsilon I$ (column $j \epsilon J$) has either two cells (one in Γ_1 and the other in Γ_2) or no cells at all.

DEFINITION 2: Let $(p, q) \epsilon B$. By *partitioning the sets I and J by* (p, q) *in B*, we mean defining $\Omega = B - \{(p, q)\}$ and determining

$$I_R = \{p\} \cup \{i \epsilon I \mid i \text{ is connected to Row } p \text{ in } \Omega\}$$

$$J_R = \{j \epsilon J \mid j \text{ is connected to Row } p \text{ in } \Omega\}$$

$$I_C = \{i \epsilon I \mid i \text{ is connected to Column } q \text{ in } \Omega\}$$

$$J_C = \{q\} \cup \{j \epsilon J \mid j \text{ is connected to Column } q \text{ in } \Omega\}$$

REMARK 3: It is easily shown that I_R and I_C (J_R and J_C) partition I (J) [10, p. 218]. The sets can be easily determined by a "scanning routine" [10, Remark 9]. If (p, q) is the unique cell in row p (column q) $I_R = \{p\}$, $J_R = \phi$ ($I_C = \phi$, $J_C = \{q\}$). It is also easily shown that by the definition of Ω, all $(i, j) \epsilon B - \{(p, q)\}$ are in $[I_R \times J_R] \cup [I_C \times J_C]$. The cell $(p, q) \epsilon [I_R \times J_C]$.

DEFINITION 3: Let Π denote a collection of cells. By a Π-solution [15, 16] we mean a solution $\{x_{ij}\}$ which satisfies

$$x_{ij} = 0 \qquad \text{for all } (i, j) \epsilon \Pi.$$

The cells $(i, j) \epsilon \Pi$ will be referred to as *inadmissible*.

DEFINITION 4: Let $F_1, F_2, \ldots, F_k$ be k different minimizing objective functions, i.e., lower values of F_i represent more desirable outcomes. A solution X is said to be *efficient* (*pareto-optimal* or *nondominated*) if and only if there does *not* exist any other solution X' which *dominates* X i.e.,

$$F_i(X') \leq F_i(X) \qquad \text{for } i = 1, 2, \ldots, k$$

with the strict inequality holding in (15) for at least one i. We will denote the above property by the condensed notation, "the solution k-tuple $(F_1, F_2, \ldots, F_k)$ is efficient."

Thus the statement that the solution pair (TC', BT') is efficient means (i) that there exists a solution X' so that $TC' = TC(X')$ and $BT' = BT(X')$, and (ii) there exists *no* solution X'' for which the corresponding $TC'' = TC(X'')$ and $BT'' = BT(X'')$ are such that either (a) $TC'' < TC'$ and $BT'' \leq BT'$ or (b) $TC'' \leq TC'$ and $BT'' < BT'$.

DETERMINATION OF EFFICIENT SOLUTION PAIRS (TC, BT).

Throughout this section we focus our attention only on the two objectives TC (Equation (4)) and BT (Equation (5)) so that when we refer to efficient solution pairs we mean only with respect to these two objectives unless otherwise specified. (The objective SB will be considered in Section 5.) A straightforward approach to obtain the efficient solution pairs is as follows: First solve the STP. Let TC^1 be the corresponding minimum total cost and BT^1 be the bottleneck time (largest t_{ij} corresponding to a strictly positive x_{ij}). Let us now declare all cells (i, j) with $t_{ij} \geq BT^1$ inadmissible and resolve the STP. If the problem has no solution then clearly the only efficient solution pair is (TC^1, BT^1). If, however, there is a solution, let TC^2 and BT^2 denote the corresponding new values.

Clearly $BT^2 < BT^1$ (by definition of BT) and $TC^2 \geq TC^1$. If $TC^2 = TC^1$ then (TC^1, BT^1) is dominated by (TC^2, BT^2) so that the latter is the only (potentially) efficient solution pair at this stage of computation. If, however, $TC^2 > TC^1$, then both can be efficient solutions. We can now continue the process by declaring all (i, j) such that $t_{ij} \geq BT^2$ as inadmissible, resolve the associated STP and continue in this manner until the STP has no solution. We state this procedure as Algorithm 1.

ALGORITHM 1: For obtaining all efficient solution pairs (TC, BT):

Step (1) Set $k=1$. Determine TC^1 by solving the standard transportation problem. Compute BT^1 using Equation (5).

Step (2) (a) Set $\Pi = \{(i, j) \epsilon [I \times J] | t_{ij} \geq BT^k\}$.

(b) If there exists no Π-solution go to step (4). Otherwise find the Π-solution that minimizes TC (an STP) and go to step (3).

Step (3) Let $k=k+1$. Let TC^k be the current optimal cost. Compute BT^k using Equation (5). Go to step (2).

Step (4) The set of potentially efficient solution pairs is $\{(TC^1, BT^1), (TC^2, BT^2), \ldots, (TC^k, BT^k)\}$.

Eliminate from this list any solution pair (TC^i, BT^i) for which there exists a solution pair (TC^j, BT^j) such that $TC^i = TC^j$ and $j > i$. Renumber the remaining l (say) solutions consecutively as $\{TC^1, BT^1), (TC^2, BT^2), \ldots, (TC^l, BT^l)\}$. This is the set of *all* efficient solution pairs (TC, BT).

JUSTIFICATION: By the definition of the set Π and Π-solution (Definition 3) it is clear that $BT^1 > BT^2 > \ldots > BT^l$. Since the set Π is strictly increasing between successive iterations, and since solutions $\{i\}$ such that $TC^i = TC^j$ for some $j > i$ are eliminated in step 4, it follows that $TC^1 < TC^2 < \ldots < TC^l$.

The justification that Algorithm 1 determines all the efficient solution pairs is now provided via Lemma 1 and Theorem 1 below.

LEMMA 1: Let $\hat{X}$ be any solution for which the corresponding values of TC and BT are $\hat{TC}$ and $\hat{BT}$. Let j be the index such that $BT^{j-1} > \hat{BT} \geq BT^j$ (where $BT^0 = \infty$, by definition). Then $\hat{TC} \geq TC^j$.

PROOF: By definition of BT^i it follows that there is no solution $\hat{X}$ for which $\hat{BT} < BT^j$. Consequently the index j can be found such that $BT^{j-1} > \hat{BT} \geq BT^j$. From step (2) and step (4) of Algorithm 1 it follows that TC^j gives the minimum cost among all Π-solutions where $\Pi = \{(i, j) \epsilon [I \times J] | t_{ij} \geq BT^{j-1}\}$. Since $\hat{BT} < BT^{j-1}$ it follows that $\hat{X}$ is a Π-solution. Since TC^j is the minimal cost Π-solution it follows that $\hat{TC} \geq TC^j$. Q.E.D.

THEOREM 1: The set $\{(TC^1, BT^1), (TC^2, BT^2), \ldots, (TC^l, BT^l)\}$ of solution pairs obtained by Algorithm 1 is the set of all efficient solution pairs and is generated in a finite number of steps.

PROOF: Consider any arbitrary solution pair $(\hat{TC}, \hat{BT})$. By Lemma 1, there exists an index $j \epsilon \{1, 2, \ldots, l\}$ such that $BT^j \leq \hat{BT}$ and $TC^j \leq \hat{TC}$. Consequently, either $(\hat{TC}, \hat{BT})$ is the same as (TC^j, BT^j) or the former is dominated by the latter. Thus the set of all efficient solution pairs is contained in $\{TC^1, BT^1), \ldots, (TC^l, BT^l)\}$. But as shown earlier for $i, j \epsilon \{1, 2, \ldots, l\}$ and $i \neq j$, $TC^i < TC^j <=> BT^i > BT^j$ so that none of these solutions dominate any other solution in this set. Consequently $\{(TC^1, BT^1), \ldots, (TC^l, BT^l)\}$ is exactly the set of all efficient solutions.

In the execution of Algorithm 1, since BT^k is strictly decreasing with k (by some finite amount), and since the $\{t_{ij}\}$ are finite and hence bounded from below, it follows that the set Π in step 2(a) will become equal to $[I \times J]$ in a finite number of steps (this will happen if and when BT^k = minimum of all $\{t_{ij}\}$). When this hapens, however, there is clearly no Π-solution and consequently the algorithm has to terminate in a finite number of steps. Q.E.D.

In Algorithm 1, the minimization of cost TC among Π-solutions can be achieved by using the Ford-Fulkerson Algorithm for solving transportation problems [4, chapter 3.1]. Assume that we have an optimum Π-solution where $\Pi = \{(i, j) \in [I \times J] | t_{ij} \geq BT^j\}$. The optimum solution has a bottleneck time of BT^{j+1}. Now to obtain the optimum Π'-solution for $\Pi' = \{(i, j) \in [I \times J] | t_{ij} \geq BT^{j+1}\}$ we have to merely set $x_{ij} = 0$ for (i, j) such that $t_{ij} = BT^{j+1}$, declare all cells (i, j) such that $t_{ij} \geq BT^{j+1}$ cells inadmissible, and continue the Ford-Fulkerson labelling procedure to obtain the new min-cost solution or show that there is no Π'-solution (i.e., the value for the maximal flow $< \sum_{j \in J} b_j$).

As an alternative to the Ford-Fulkerson procedure, we present below an alternate algorithm, which starting with the optimal Π-solution obtains the optimal Π'-solution (the Π and Π' sets are as defined in the earlier paragraph) by gradually driving each x_{ij} to zero for (i, j) such that $t_{ij} = BT^{j+1}$ and $x_{ij} > 0$ while simultaneously maintaining optimality in terms of TC. (The method for reducing x_{ij} to 0 will be provided later.) While both procedures are likely to be approximately the same in terms of computational efficiency, the alternate algorithm to be presented has some additional advantages: (a) it easily extends to the consideration of the third objective of minimizing SB, and (b) it is interesting in that, for the special case $c_{ij} = 0$ for all (i, j), it reduces to the Szwarc-Hammer algorithm for the BTP.

ALGORITHM 2: For obtaining all efficient solutions (TC, BT): This algorithm is identical to Algorithm 1 except that step 2 is replaced by 2′ below:

Step (2′) (a) Set $\Pi = \{(i, j) \in [I \times J] | t_{ij} \geq BT^k$ and $x_{ij} = 0\}$

(b) Choose a cell (p, q) such that $t_{pq} = BT^k$ and $x_{pq} > 0$. If there is no such (p, q) go to step 3. Otherwise go to step 2′ (c).

(c) Drive x_{pq} down to zero while maintaining TC-optimality among Π-solutions. If x_{pq} cannot be driven to zero, go to step 4. Otherwise set $\Pi = \Pi \cup \{p, q)\}$ and go to step 2′ (b).

JUSTIFICATION: When we arrive at step 2(a) of Algorithm 1 or step 2′(a) of Algorithm 2, the current bottleneck time is BT^k. The set of cells Π in step 2(a) of Algorithm 1 can be partitioned into three subsets:

(i) $\Pi_1 = (i, j)$ such that $t_{ij} > BT^k$. By this definition of BT^k, $x_{ij} = 0$ for these cells (this subset may be empty when $k = 1$)

(ii) $\Pi_2 = (i, j)$ such that $t_{ij} = BT^k$ but $x_{ij} = 0$ (this subset may be empty) and

(iii) $\Pi_3 = (i, j)$ such that $t_{ij} = BT^k$ and $x_{ij} > 0$ (this subset is never empty at step 2(a) or 2′(a)).

Step 2′(a) of Algorithm 2 defines Π as $\Pi_1 \cup \Pi_2$. Then steps 2′(b)–(c) drive each of the cells in Π_2 to zero. Soon after a cell (p, q) is driven to zero (while maintaining TC-optimality among Π-solutions), it drops out of the set Π_3 and is added to set Π_2 (and hence Π). When $\Pi_3 = \phi$, then we have an optimal Π-solution where $\Pi = \{(i, j) \in (I \times J) | t_{ij} \geq BT^k\}$ as defined in step 2(a) of Algorithm 1. However, some x_{pq} for $(p, q) \in \Pi_3$ cannot be driven to zero, this clearly means that there is no such Π-solution.

The rule for choosing a $(p, q) \in \Pi_3$ in step 2′(b) could potentially affect the computational p formance of Algorithm 2. However, our experience with a very similar algorithm [14, Section shows that all the four choice rules given below perform approximately equally in terms of comput tional efficiency:

(i) choose a $(p, q) \in \Pi_3$ with the largest x_{pq}
(ii) choose a $(p, q) \in \Pi_3$ with the smallest x_{pq}
(iii) choose a $(p, q) \in \Pi_3$ if it is the unique basis cell in its row or column; if no such (p, q) exis use rule (i),
(iv) same as rule (iii) except for default rule (ii) is used instead of (i).

Rule (i) is used with the hope of proving infeasibility quickly, rule (ii) is used to speed up t driving of (p, q) out of Π_3 into Π_2. The reason for the first part of rules (iii) and (iv) is that, as w be seen later, the computational effort of driving x_{pq} down to zero is likely to be somewhat simpl when (p, q) is the unique basis cell in its row or column. In Algorithm 3 to be presented later, use rule (i). (If there is more than one $(p, q) \in \Pi_3$ with the same maximum x_{pq}, such ties may broken arbitrarily.)

We now consider an operational procedure for driving x_{pq} down while maintaining *TC*-opt mality among Π-solutions. This can be formulated as a two-objective problem, where

$$F_1=\sum_{i\in I}\sum_{j\in J} c_{ij}x_{ij} \tag{16}$$

and

$$F_2=x_{pq} \tag{17}$$

where again, lower values of either objective are more desirable *ceteris paribus*, and we consider on Π-solutions. Denoting by F'_1 and F'_2 the values for F_1 and F_2 when we start with a cell (p, q) step 2′(b), the problem of driving x_{pq} to zero while maintaining *TC*-optimality among Π-solutio can be thought of as obtaining the set of efficient solution pairs (F_1, F_2) starting with $(F'_1, F'$ and ending (or, at least trying to end) in (F''_1, F''_2) where $F'_2>0$ and $F''_2=0$ (obviously, F''_1 F'_1). It should be emphasized that this two-objective problem is only a "subroutine" (althoug important computationally) for obtaining all efficient solution pairs (*TC*, *BT*).

Let us now consider the compound objective

$$G=\alpha F_1+(1-\alpha)F_2 \qquad \text{where } 0<\alpha<1. \tag{18}$$

It is well known (e.g., [6]) that minimizing G leads to an efficient solution pair $(F_1(X), F_2(X))$. T see this obvious result, assume the contrary, i.e., an optimal Π-solution X to (18) is not efficien Then by Definition 4, it follows that there exists a Π-solution X' for which $F'_1\leq F_1$ and $F'_2<F_2$ $F'_1<F_1$ and $F'_2\leq F_2$. In either case, $G'=\alpha F'_1+(1-\alpha)F'_2<\alpha F_1+(1-\alpha)F_2=G$, contradicting t assumption that X is an optimal Π-solution to (18). Remark 7.4.1 of [9] shows that the converse also true, i.e., for every efficient X there exists a scalar $\alpha(0<\alpha<1)$ such that X is an optimal solutio to (18). (This is so, since we have only two linear objectives.) Since $0<\alpha<1$, we can define $(1-\alpha)/\alpha=$ so that instead of minimizing (18) we can equivalently minimize $(F_1+\delta F_2)$ so that the followin Theorem holds:

THEOREM 2: For $\delta>0$ every optimal solution to (19) below is efficient in terms of F_1 and F_2.

(19) $$\text{Minimize } H=F_1+\delta F_2=\sum_{i\in I}\sum_{j\in J} c_{ij}x_{ij}+\delta x_{pq}.$$

Conversely, if X is efficient, there exists a $\delta>0$ such that X is an optimal solution to (19).

We now prove a result with respect to the minimum value for the second objective F_2.

THEOREM 3: Among all Π-solutions, the solution X corresponding to the minimum value of F_2 will be obtained as an optimal solution to (19) in the limit as $\delta\to\infty$.

PROOF: The objective F_2 is bounded below by zero since we restrict our attention to solutions. Since minimizing F_2 over Π-solutions is a linear programming problem, it is clear that the minimum value of F_2 exists. Let F^*_2 be the limiting value of F_2 obtained as $\delta\to\infty$. Let F^*_1 be the corresponding value for F_1. Thus $(F^*_1+\delta F^*_2)$ is the minimum of (19) as $\delta\to\infty$. Now assume the contrary, i.e., there exists a Π-solution X' for which the corresponding $F'_2<F^*_2$. Let F'_1 be the value of F_1 corresponding to X'. Since (F^*_1, F^*_2) is efficient (Theorem 2) and since $F'_2<F^*_2$ was assumed, it follows that $F'_1>F^*_1$. Let $\hat{\delta}$ be the value of δ such that $F'_1+\hat{\delta}F'_2=F^*_1+\hat{\delta}F^*_2$ or $\hat{\delta}=(F'_1-F^*_1)/(F^*_2-F'_2)$. Clearly $\hat{\delta}>0$. Now for $\delta>\hat{\delta}$ it is easily seen that $F'_1+\delta F'_2<F^*_1+\delta F^*_2$ contradicting the fact that $(F^*_1+\delta F^*_2)$ is the minimum of (19) as $\delta\to\infty$. Q.E.D.

The problem of minimizing (19) among Π-solutions can be formulated as a standard transportation problem P^+ by suitably defining the cost matrix $C^+=\{c_{ij}{}^+\}$ as follows:

(20) $$c_{ij}{}^+=\begin{cases}\infty & \text{for } (i,j)\in\Pi'\\ c_{ij}+\delta & \text{for } (i,j)=(p,q)\\ c_{ij} & \text{elsewhere}\end{cases}$$

$$\text{where } \Pi'=\{(i,j)\in[I\times J]\,|\,t_{ij}\geq BT^k \text{ and } (i,j)\notin B\}.$$

Since $c_{ij}{}^+=\infty$ for $(i,j)\in\Pi'$, the objective of minimizing the total cost formulated as a standard transportation problem would maintain $x_{ij}=0$ for $(i,j)\in\Pi'$. The set Π′ used in (20) is slightly different from the set Π used in step 2′(a) of Algorithm 2 in that the statement $x_{ij}=0$ is replaced by $(i,j)\notin B$. Now $(i,j)\notin B\rightarrow x_{ij}=0$ but the converse is not necessarily true (i.e., x_{ij} can be zero for a basic cell). Thus (20) leaves out of the set Π′ those basic cells which have $x_{ij}=0$ and $t_{ij}\geq BT^k$. However, as will be seen later, x_{ij} will be maintained at zero even for these cells (i.e., Π−Π′), which is, after all, the only requirement of a Π-solution.

Let us now assume that we have the basic optimal solution with primal solution $\{x_{ij}\}$ and duals $\{u_i\}$ and $\{v_j\}$ (see the notations and definitions in Section 2) to the problem P, i.e., problem P when $\delta=0$. The optimal solutions to P^+ as δ is parametrically increased from 0 to ∞ can be obtained from the optimal solution of P via the (*plus*) *cell cost operator* $\delta\boldsymbol{C}^+{}_{pq}$ discussed in [10, 11]. Since the present problem has no upper bounds on the $\{x_{ij}\}$, and since we are interested only in the special case when $(p,q)\in B$ (since $x_{pq}>0$ in step 2′(b) of Algorithm 2, it follows that (p,q) is a basic cell), we can simplify somewhat the results in [10, 11]. (Hence the reader need not refer to those references.) By way of definition, the cell cost operator $\delta\boldsymbol{C}^+{}_{pq}$ transforms the optimal solutions $\{v_j\},\{u_i\}$ and $\{v_j\}$ for P to the optimal solutions $\{x^+{}_{ij}\},\{u^+{}_i\}$ and $\{v^+{}_j\}$ for the problem P^+ whose data (i.e., $\{c_{ij}\}$, $\{a_i\}$ and $\{b_j\}$) are the same as those of P except $c^+{}_{pq}=c_{pq}+\delta(\delta\geq 0.)$ (Note, however, that the $\{c_{ij}\}$ now denote the unit costs for $\delta=0$. Thus from (2), $c_{ij}=\infty$ for $(i,j)\in\Pi'$ and c_{ij} elsewhere.) The operator is said to be *basis preserving*, and denoted by light face letters $\delta C^+{}_{pq}$ if the optimal solution to P^+ has the same basis as that for P; i.e., $B^+=B$. We denote by μ the maximum

value for δ so that the operator is basis preserving and will denote by H and H^+ the optimal value of the objective function (19) for P and P^+.

THEOREM 4 (in part, adapted from Theorems 5(a) and 7(a) in [10]): For the basis preserving cell cost operator δC^+_{pq}, the primal solution is given by

$$x^+_{ij}=x_{ij} \quad \text{for } i\epsilon I \text{ and } j\epsilon J. \tag{21}$$

The dual solutions are determined by first partitioning sets I and J by (p, q) in B (See Definition 2 in Section 2) and setting

$$u^+_i=\begin{cases} u_i+\delta & \text{for } i\epsilon I_R \\ u_i & \text{for } i\epsilon I_C \end{cases} \tag{22}$$

$$v_j^+=\begin{cases} v_j-\delta & \text{for } j\epsilon J_R \\ v_j & \text{for } j\epsilon J_C. \end{cases} \tag{23}$$

The optimal value $H^+=H+\delta x_{pq}$ and the maximum extent μ is given by

$$\mu=\underset{(i,j)\epsilon\psi}{\text{Min}}\ (c_{ij}-u_i-v_j) \tag{24}$$

where

$$\psi=[(I_R\times J_C)-\{(p,q)\}]\cap\tilde{\Pi}', \tag{25}$$

where $\tilde{\Pi}'$ is the complement of the set Π' previously defined.

If the set ψ is empty, then the current value of x_{pq} is the lowest attainable value for the objective $F_2=x_{pq}$.

PROOF: Since the operator is basis preserving (i.e., $B^+=B$) and since the $\{a_i\}$ and $\{b_j\}$ have not changed, Equation (21) holds. From (19) it then readily follows that $H^+=H+\delta x_{pq}$. From (22)–(23) it follows that $u_i^++v_j^+=u_i+v_j$ for $(i, j)\ \epsilon\ [I_R \text{ x } J_R]\ \cup\ [I_C \text{ x } J_C]$. From Remark 3, all the basic cells in $B^+(=B)$ except (p, q) are in $[I_R \text{ x } J_R]\ \cup\ [I_C \text{ x } J_C]$. Since $c_{ij}^+=c_{ij}$ for $(i, j)\neq(p, q)$ Equation (8) is satisfied by these basis cells for P^+. By Remark 3, the basic cell $(p, q)\ \epsilon\ [I_R \text{ x } J_C]$ so that $c_{pq}^+=c_{pq}+\delta=u_p+v_q+\delta=u_p^++v_q^+$. Thus the solutions (22)–(23) satisfy (8) for problem P^+.

For the cells $(i, j)\ \epsilon\ [I_R \text{ x } J_R]\ \cup\ [I_C \text{ x } J_C]$ we note that $c_{ij}^+=c_{ij}$, $u_i^++v_j^+=u_i+v_j$ so that (9) is satisfied by all such cells. For $(i, j)\ \epsilon\ [I_C\times J_R]$, $u_i^++v_j^+=u_i+v_j-\delta$ so that (9) is satisfied readily for P^+ since it is satisfied for P and since $c_{ij}^+=c_{ij}$ for these cells. For $(i, j)\ \epsilon\ [I_R\times J_C]$, however, $u_i^++v_j^+=u_i+v_j+\delta$. For the cells $(i, j)\ \epsilon\ \Pi'$ the condition (9) is automatically satisfied since $c_{ij}=\infty$ for these cells. For the remaining nonbasic cells in $[I_R\times J_C]$ (note that $(p, q)\ \epsilon\ B$), the condition (9) can be written as

$$c_{ij}^+=c_{ij}\geq u_i+v_j+\delta \text{ for } (i, j)\ \epsilon\ [I_R\times J_C],\ (i, j)\neq(p, q),\ (i, j)\ \epsilon\ \Pi'.$$

Since $\delta\leq\mu$ for the basis preserving operator, Equations (24)–(25) assure us that the dual feasibility condition (9) is maintained for P^+. Since (21)–(23) define a pair of primal and dual feasibile solutions to P^+, these are also optimal.

Now suppose the set $\psi=\phi$. Then by the preceding arguments it follows that the solutions (21)–(23) define an optimal solution to P^+ for all $\delta\geq 0$. Consequently as $\delta\rightarrow\infty$, this solution is still optimal and hence by Theorem 3, it follows that $\psi=\phi\Longrightarrow$ the minimum value for x_{pq} has been attained. Q.E.D.

The basis-preserving operator provides the transformed optimal solutions only for $\delta\leq\mu$. To obtain the optimal solution for $\delta>\mu$, a sequence of basis-preserving operators need to be applied [11, Section 4]. More generally, at the lth iteration of driving out x_{pq} (the l appears as a subscript

t distinguish it from the index k for efficient point (TC^k, BT^k) which appears as a superscript), l μ_l be the maximum extent to which $\delta C_{pq}{}^+$ can be applied to the problem P_l with basis B_l. The mount μ_l was determined by Equation (24) such that if $\delta > \mu_l$ some cell $(e, f) \in \psi$ would violate the al feasibility condition (9). (Cell (e, f) is the one at which the minimum of (24) is attained. The scripts l on (e_l, f_l) have been dropped to avoid confusing notation later.) Let P_{l+1} be the problem t at results on applying $\mu_l C_{pq}{}^+$ to P_l. By definition of μ_l and the cell (e, f), it follows that

(i) $$u_{e,\,l+1} + v_{f,\,l+1} = c_{ef}.$$

Thus (e, f) satisfies (8) and hence qualifies as a basic cell for B_{l+1}. If we add (e, f) to B_l a cycle will t created with $(e, f) \in \Gamma_2$. Let us obtain the *alternate optimal solution* (because of (26), introducing (f) into the basis will not alter the value of the objective function H)

(') $$x_{ij.\ l+1} = \begin{cases} x_{ij.\ l} - \Delta & \text{for } (i, j) \in \Gamma_1 \\ x_{ij.\ l} + \Delta & \text{for } (i, j) \in \Gamma_2 \\ x_{ij.\ l} & \text{elsewhere} \end{cases}$$

where

(3) $$\Delta = \operatorname*{Min}_{(i.\ j) \in \Gamma_1} (x_{ij.\ l}).$$

In (27) we subtract Δ from the odd parity cells ("givers") and add Δ to the even parity cells (getters"). The amount Δ is determined in (28) as the minimum $x_{ij,l}$ of the "givers". Let $(r, s) \in \Gamma_1$ t the cell at which this minimum is taken so that $x_{rs,\,l+1} = 0$. Then $B_{l+1} = B_l - \{(r, s)\} + \{(e, f)\}$ defines a basis (the cell $(r, s) \notin B_{l+1}$ satisfies $x_{rs,\,l+1} = 0$), and the solution $\{x_{ij,\,l+1}\}$ defines a basic otimal solution to P_{l+1}. (By the choice of Δ in (28), $x_{ij,\,l+1} \geq 0$.) Further, from Remark 2 and Equation (27) it follows that $\{x_{ij,\,l+1}\}$ satisfies conditions (1) and (2) just as $\{x_{ij,\,l}\}$. The duals $\{u_{i,\,l+1}\}$ and $\{v_{j,\,l+1}\}$ are not altered by this basis change since (e, f) satisfies (8) and hence the dual feasibility conditions (8) and (9) continue to be satisfied.

Having obtained P_{l+1} with basis B_{l+1} and solutions $\{x_{ij,\,l+1}\}$, $\{u_{i,\,l+1}\}$, and $\{v_{j,\,l+1}\}$ we can again use Equation (24) to determine μ_{l+1}, and obtain P_{l+2} by applying $\mu_{l+1} C^+{}_{pq}$ to P_{l+1} to obtain the appropriate $\{u_{i,\,l+2}\}$ and $\{v_{j,\,l+2}\}$. Then the basis B_{l+1} can be changed to B_{l+2} as before and the new $\{v_{j,\,l+2}\}$ computed. The procedure can be repeated for $l+3$, ... until (i) either $(p, q) = (r, s)$ so that (p, q) leaves the basis and hence $x_{pq} = 0$, or (ii) the set ψ (25) becomes empty so that the corresponding $\mu = \infty$ and hence the minimum value of x_{pq} has been attained

Theorem 5 below shows that during the transformation (27), the value of x_{pq} decreases by Δ. (cf. Lemma 6(a) in [11]).

THEOREM 5: During the transformation of solution from X_l to X_{l+1} outlined earlier

(5) $$x_{pq.\ l+1} = x_{pq.\ l} - \Delta.$$

PROOF: From (27), it follows that the proof will be complete if we can show that $(p, q) \in \Gamma_1$. The cycle Γ is created by the addition of the cell (e, f) to B_k. The cell (e, f) was determined as the cell at which the minimum of (24) was attained so that $(e, f) \in [I_R \times J_C]$ so that $e \in I_R$, $p \in I_R$, $f \in J_C$, and $q \in J_C$. Since (p, q) is the only basis cell in $[I_R \times J_C]$, the cycle Γ will have to involve (p, q), and, in fact, Γ can be explicitly written as

(6) $$\Gamma = \{(e, f)\} \cup \{S_1 = \text{path in } B \text{ from column } f \text{ to column } q\} \cup \{(p, q)\} \cup \{S_2 = \text{path in } B \text{ from row } p \text{ to row } e\}.$$

The set $S_1 \subseteq [I_C \times J_C]$ and is empty if $f=q$. Similarly $S_2 \subseteq [I_R \times J_R]$ and is empty if $p=e$. B Remark 1 the number of cells in S_1 is even, so that starting the counting from (e, f) as 0, $(p,$ will occupy an odd-numbered position in Γ and hence $(p, q) \in \Gamma_1$. Q.E.I

When the solution X_l was transformed to X_{l+1}, this was described as an alternate optimu since the nonbasic cell (e, f) which is brought into the basis has zero "reduced cost" (see (26) Consequently the objective $H_l = H_{l+1}$. From (19), it then follows that $F_{1,l} + \delta F_{2,l} = F_{1,l+1} + \delta F_{2,l+}$ But from Theorem 5, $F_{2,l+1} = x_{pq,l+1} = x_{pq,l} - \Delta = F_{2,l} - \Delta$ so that

$$F_{1,l+1} = F_{1,l} + \delta\Delta. \tag{31}$$

Denoting by TC_l and TC_{l+1} the total cost objective F_1, it then follows that

$$TC_{l+1} = TC_l + \delta\Delta. \tag{32}$$

The transformation (27) of the solution X_l may lead to a solution X_{l+1} which is *not* a Π-solu tion. This is because of the way in which the set Π is defined in Equation 2′(a). The set Π in ste 2′(a) of Algorithm 2 can include a cell $(i, j) \in B$ with $t_{ij} \geq BT^k$ but with $x_{ij} = 0$. As remarked earlie these cells are not in Π' (Equation (20)). If such a cell $(i, j) \in \Gamma_1$ (a "giver"), then $\Delta = 0$ (28) an hence (i, j) can be chosen as the cell (r, s) to leave the basis. If, however, $(i, j) \in \Gamma_2$ (a "getter" then $x_{ij,l+1}$ may increase. To avoid this possibility, whenever the cycle Γ includes a $(\Pi - \Pi')$ cel (i.e., $(i, j) \in B$, $x_{ij} = 0$ and $t_{ij} \geq BT^k$)(r, s) in Γ_2, then we set $X_{l+1} = X_l$ and $B_{l+1} = B_l + \{(e, f)\} -$ $\{(r, s)\}$ (cf. step 4(a) of the *TTP* method [16]). We can now apply $\delta C_{pq}{}^+$ to B_{l+1} and continue th iterations.

Algorithm 3 below gives a detailed procedure for obtaining all efficient solution pairs (TC, BT) It is the same as Algorithm 2 but with an explicit description of step 2′(c) for driving x_{pq} down t zero while maintaining TC-optimality among Π-solutions.

ALGORITHM 3: For obtaining all efficient solution pairs (TC, BT):

Step (1) Set $k=1$. Let $C^1 = C$ be the matrix of cost coefficients. Let $T = \{t_{ij}\}$ be the matrix of and $\{a_i\}$ and $\{b_j\}$ the warehouse availabilities for $i \in I$ and market requirements for $j \in J$ respectively. Solve the standard transportation problem of minimizing total cost. Let B be the optimal basis, $X = \{x_{ij}\}$ be the optimal primal solution, $\{u_i\}$ and $\{v_j\}$ the optimal dual variables. Let the cost of this optimal solution be TC^1.

Step (2) Initialize $TC^{k+1} = TC^k$ and compute the Bottleneck Time BT^k using (5). Derive the cost matrix C^{k+1} from C^k as follows:

$$c_{ij}^{k+1} = \begin{cases} M & \text{if } t_{ij} \geq BT^k \text{ and } (i, j) \notin B. \\ c_{ij}{}^k & \text{otherwise} \end{cases}$$

where M is a very large positive number (e.g. 10^{30}).

Step (3) Choose a cell $(p, q) \in B$ for which $x_{pq} > 0$ and $t_{pq} = BT^k$. If there is no such cell set $k = k+1$ and go to step (2). If there is more than one such (p, q) choose the cell with the largest x_{pq} (Break ties in any arbitrary way). Let $\delta = 0$ and $l = 1$. Denote by $\{x_{ij,1}\}$, $\{u_{i,1}\}$ and $\{v_{j,1}\}$ the current values of $\{x_{ij}\}$, $\{u_i\}$ and $\{v_j\}$. Let B_1 denote the current basis.

(4) Determine I_R, J_R, I_C, J_C corresponding to (p, q) and B_l. Set

$$\psi=\{(i, j)\epsilon[I_R\times J_C]-\{(p, q)\} \text{ such that } c_{ij}^{k+1}\neq M\}$$

If ψ is empty go to step (10). Otherwise compute

$$\mu_l=\operatorname*{Min}_{(i,j)\epsilon\psi}(c_{ij}^{k+1}-u_{i,l}-v_{j,l})$$

and record the cell (e, f) where the minimum is taken.

(5) Set

$$u_{i,l+1}=\begin{cases}u_{i,l}+\mu_l & \text{for } i\epsilon I_R\\ u_{i,l} & \text{for } i\epsilon I_C\end{cases}$$

and

$$v_{j,l+1}=\begin{cases}v_{j,l}-\mu_l & \text{for } j\epsilon J_R\\ v_{j,l} & \text{for } j\epsilon J_C\end{cases}$$

Redefine c_{pq}^{k+1} as $c_{pq}^{k+1}+\mu_l$. Set $\delta=\delta+\mu_l$.

(6) Determine the cycle Γ created by the addition of (e, f) to B_l. If there is an $(r, s)\ \epsilon\ \Gamma_2$ with $x_{rs,l}=0$ and $t_{rs}\geq BT^k$ go to step (9). Otherwise go to step (7).

(7) Let $\chi=\{(g, h)\}$ be the cells such that $x_{gh,l}$ is the minimum of $(x_{ij,l})$ for $(i, j)\ \epsilon\ \Gamma_1$. If $(p, q)\ \epsilon\ \chi$ choose $(r, s)=(p, q)$. Otherwise choose some (r, s) in χ arbitrarily. Let $\Delta=x_{rs,l}$. Set

$$x_{ij,v+1}=\begin{cases}x_{ij,l}-\Delta & \text{for } (i, j)\epsilon\Gamma_1\\ x_{ij,l}+\Delta & \text{for } (i, j)\epsilon\Gamma_2\\ x_{ij,l} & \text{elsewhere.}\end{cases}$$

(8) If $t_{rs}\geq BT^k$ redefine $c_{rs}^{k+1}=M$. Redefine $TC^{k+1}=TC^{k+1}+\delta\Delta$. Set $B_{l+1}=B_l+\{(e, f)\}-\{(r, s)\}$. If $(p, q)=(r, s)$ set $B=B_{l+1}$ and go to step (3). Otherwise set $l=l+1$ and go to step (4).

(9) Set $X_{l+1}=X_l$. Redefine $c_{rs}^{k+1}=M$. Set $B_{l+1}=B_l+\{(e, f)\}-\{(r, s)\}$, $l=l+1$ and go to step (4).

(10) The set of potentially efficient solution pairs is

$$\{(TC^1, BT^1), (TC^2, BT^2), \ldots, (TC^k, BT^k)\}.$$

Eliminate from this list any solution (TC^i, BT^i) for which there exists a (TC^j, BT^j) such that $TC^i=TC^j$ and $j>i$. Renumber the remaining l (say) solutions consecutively as $\{(TC^1, BT^1), (TC^2, BT^2), \ldots, (TC^l, BT^l)\}$. This is the set of all efficient solutions (TC, BT).

Theorem 6 below provides the convergence proof for Algorithm 3.

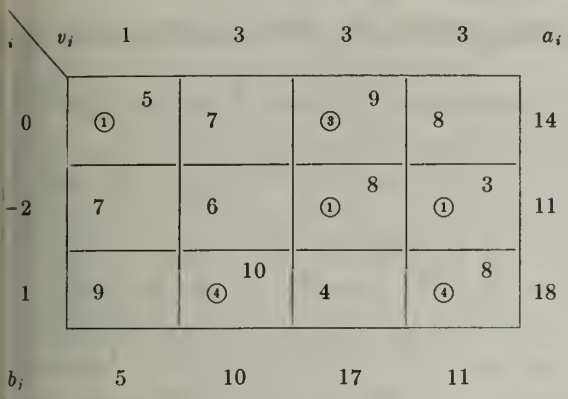
THEOREM 6: Algorithm 3 determines the set of all efficient solution pairs in a finite number of steps.

PROOF: Algorithm 2 (and hence Algorithm 3) differs from Algorithm 1 only in the way step 2 of Algorithm 1 is executed. As shown in Theorems 2–5 and the related exposition, steps (2)–(9) of Algorithm 3 obtain a Π-solution where $\Pi=\{(i, j)\epsilon[I\times J]|t_{ij}\geq BT^k\}$ (or demonstrate the nonexistence of a Π-solution) by driving each x_{pq} to 0 (for (p, q) such that $x_{pq}>0$ and $t_{pq}=BT^k$) while maintaining TC-optimality among Π-solutions. That the cell cost operator δC^+_{pq} can be applied

in a finite number of steps as $\delta \to \infty$ is proved as Theorem 1 in [14]. (This proof applies even if t problem is degenerate. The only potential complication in Algorithm 3 is that caused by step (But this too causes no problem, since when (r, s) is dropped its cost is reset to M so that (r, s) w never become a basis cells again. Consequently there is no possibility of cycling or circling Therefore, from Theorem 1 it follows that Algorithm 3 determines the set of all efficient solutio in a finite number of steps. Q.E.

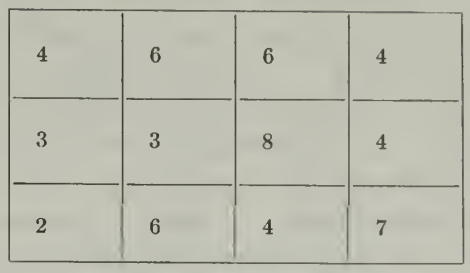
EXAMPLE: Figure 1(a) shows the tableau for a (3×4) transportation problem with cos $C=\{c_{ij}\}$, availabilities $\{a_i\}$ and requirements $\{b_j\}$. The circled cells in Figure 1(a) constitute t optimal basis B for the standard transportation problem of minimizing total cost. The amoun $\{x_{ij}\}$ for $(i, j)\in B$ are shown above the circled cells ($x_{ij}=0$ for $(i, j)\notin B$.) The optimal dual variabl $\{u_i\}$ and $\{v_j\}$ satisfying (8)–(9) are also shown in Figure 1(a). The optimal cost $TC^1=115$. T times $\{t_{ij}\}$ are displayed in Figure 1(b). We now indicate step by step the determination of the ef cient solution pairs $\{(TC, BT)\}$ using Algorithm 3.

Step	Result
(1)	$k=1$, $TC^1=115$.
(2)	$TC^2=115$, $BT^1=8$, Matrix C^2 shown in Figure 1(c).
(3)	$(p, q)=(2, 3)$. $\delta=0$, $l=1$. The solution $X_1=X$ with basis $B_1=B$ and solutions $\{u_{i,1}=u_i\}$ and $\{v_{i,1}=\}$ are shown in Figure 1(c).
(4)	The set of rows that are connected to row 2 in B_1 without using (2, 3) are $I_R=\{2, 3\}$. Similarly the se columns connected to column 3 in B_1 without using (2, 3) are $J_C=\{1, 3\}$ so that $\psi=\{(2, 1), (3, 1), (3, 3$ $\mu_1=0$ and $(e, f)=(3, 3)$.
(5)	$\{u_{i,2}\}$, $\{v_{j,2}\}$ shown in Figure 1(d). Since $\mu_1=0$, c_{23}^2 remains at 1 and $\delta=0$.
(6)	The cycle Γ is shown in Figure 1(c) as $\{(3, 3), (3, 4), (2, 4), (2, 3)\}$ so that $\Gamma_1=\{(3, 4), \{2, 3)\}$.
(7)	$\chi=\{(3, 4), (2, 3)\}$. Thus $(r, s)=(2, 3)$. $\Delta=8$. $X_2=\{x_{ij,2}\}$ shown in Figure 1(d).
(8)	The cost c_{23}^2 is set equal to M (Figure 1(d)). $TC^2=115$. $B_2=B_1+\{(3, 3)\}-\{(2, 3)\}$ (Figure 1($(p, q)=(r, s)$, $B=B_2$.
(3)	No (p, q) such that $x_{pq}>0$ and $t_{pq}=8$. $k=2$.
(2)	$TC^3=115$, $BT^2=6$. C^3 shown in Figure 1(e). Note that $c_{12}^3=M$ but $c_{34}^3\neq M$.
(3)	$(p, q)=(3, 2)$, $\delta=0$, $l=1$. Figure 1(e) also shows B_1, X_1, $\{u_{i,1}\}$ and $\{v_{j,1}\}$.
(4)	$I_R=\{1, 2, 3\}$ and $J_C=\{2\}$ (Remark 3, Section 2). $\psi=(e, f)=\{(2, 2)\}$. $\mu_1=5$.
(5)	$\{u_{i,2}\}$ and $\{v_{j,2}\}$ shown in Figure 1(f). c_{32}^3 is increased to 9 and $\delta=5$.
(6)	$\Gamma=\{(2, 2), (3, 2), (3, 4), (2, 4)\}$ (Figure 1(e)). $(3, 4)\in\Gamma_2$ with $x_{34}=0$ and $t_{34}=7>BT^2=6$.
(9)	$X_2=X_1$, $c_{34}^3=M$ and $B_2=B_1+\{(2, 2)\}-\{(3, 4)\}$ as shown in Figure 1(f). $l=2$.
(4)	$I_R=\{1, 3\}$, $J_C=\{2, 4\}$. $\psi=(e, f)=\{(1, 4)\}$. $\mu_2=5$.
(5)	$\{u_{i,3}\}$, $\{v_{j,3}\}$ and $c_{32}^3=9+5=14$ shown in Figure 1(g). $\delta=10$.
(6)	Γ shown in Figure 1(f).
(7)	$\chi=(r, s)=\{(1, 3)\}$. $\Delta=9$. X_3 is shown in Figure 1(g).
(8)	c_{13}^3 is set equal to M (Figure 1 (g)). $TC^3=115+(10\times9)=205$. $B_3=B_2+\{(1, 4)\}-\{(1, 3)\}$ shown Figure 1(g). $l=3$.
(4)	$I_R=\{3\}$. $J_C=\{1, 2, 4\}$. $\psi=(e, f)=\{(3, 1)\}$, $\mu_3=7$.
(5)	$\{u_{i,4}\}$, $\{v_{j,4}\}$ are shown in Figure 1(h). $c_{32}^3=21$. $\delta=17$.
(6)	Γ shown in Figure 1(g). $\chi=(r, s)=(3, 2)$.
(7)	$\Delta=1$. X_4 shown in Figure 1(h).
(8)	$c_{32}^3=M$ (Figure 1(h)). $TC^3=205+(17\times1)=222$. $B_4=B_3+\{(3, 1)\}-\{(3, 2)\}$ shown in Figure 1($(p, q)=(r, s)$, $B=B^4$.
(3)	No $(p, q)\in B$ for which $x_{pq}>0$ and $t_{pq}=6$. $k=3$.
(2)	$TC^4=222$. $BT^3=4$. C^4 same as C^3 as shown in Figure 1(h).
(3)	$(p, q)=(3, 3)$. $\delta=0$, $l=1$. B_1, X_1, $\{u_{i,1}\}$ and $\{v_{j,1}\}$ also shown in Figure 1(h).
(4)	$I_R=\{1, 2, 3\}$, $J_C=\{3\}$. $\psi=\phi$.
(10)	The set of potential efficient solution pairs is $\{(115, 8), (115, 6), (222, 4)\}$. These are shown as points A, B d E in Figure 1(i). (The intermediate calculations are shown as points C and D). We eliminate (115, 8) fm this list. Consequently the set of all efficient pairs is $\{(115, 6), (222, 4)\}$ (points B and E).



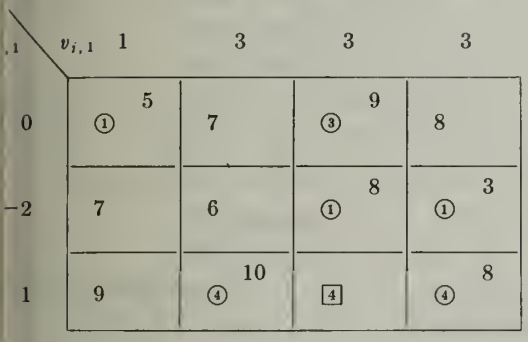
$C^1 = C$, B, X shown. $TC^1 = 115$

Fig. 1(a)



Matrix $T = \{t_{ij}\}$

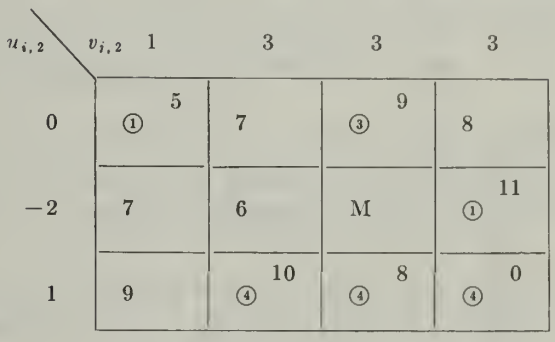
Fig. 1(b)



$k=1$, $TC^2=115$, $BT^1=8$, $(p, q)=(2, 3)$

$l=1$, $\delta=0$, $(C^2, B_1, X_1,)$ shown

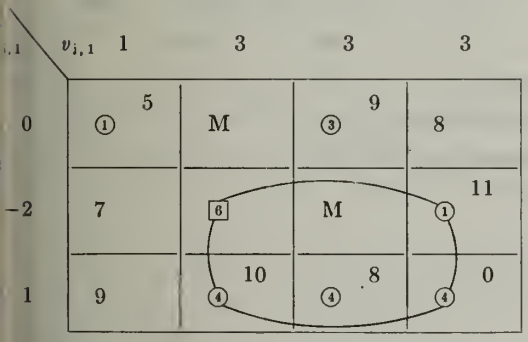
Fig. 1(c)



$k=1$, $TC^2=115$, $(p, q)=(2, 3)$

$\delta=0$, (C^2, B_2, X_2) shown

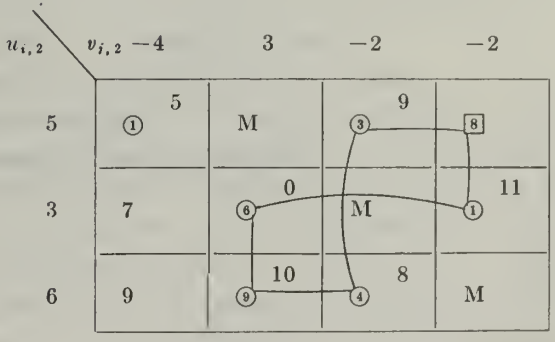
Fig. 1(d)



$k=1$, $TC^3=115$, $BT^2=6$, $(p, q)=(3, 2)$

$l=1$, $\delta=0$, (C^3, B_1, X_1) shown

Fig. 1(e)

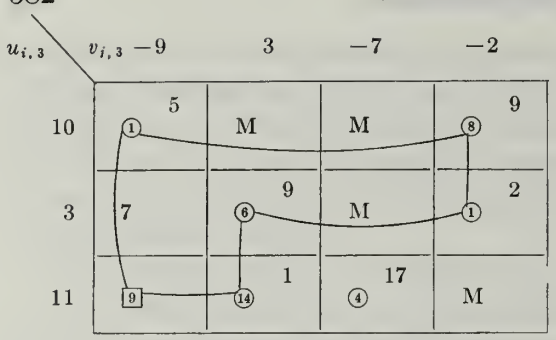


$k=2$, $TC^3=115$, $(p, q)=(3, 2)$

$l=2$, $\delta=5$, (C^3, B_2, X_2) shown

Fig. 1(f)

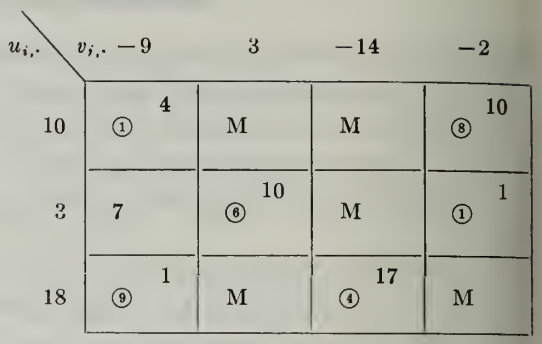
Figure 1. Computing efficient pairs (TC, BT)



$k=2$, $TC^3=205$, $(p, q)=(3, 2)$

$l=3$, $\delta=10$, (C^3, B_3, X_3) shown

Fig. 1(g)



$k=2$, $TC^3=222$, $(p, q)=(3, 2)$

$\delta=17$, (C^3, B_4, X_4) shown

$k=3$, $TC^4=222$, $BT^3=4$, $(p, q)=(3, 3)$

$l=1$, $\delta=0$, (C^4, B_1, X_1)

Fig. 1(h)

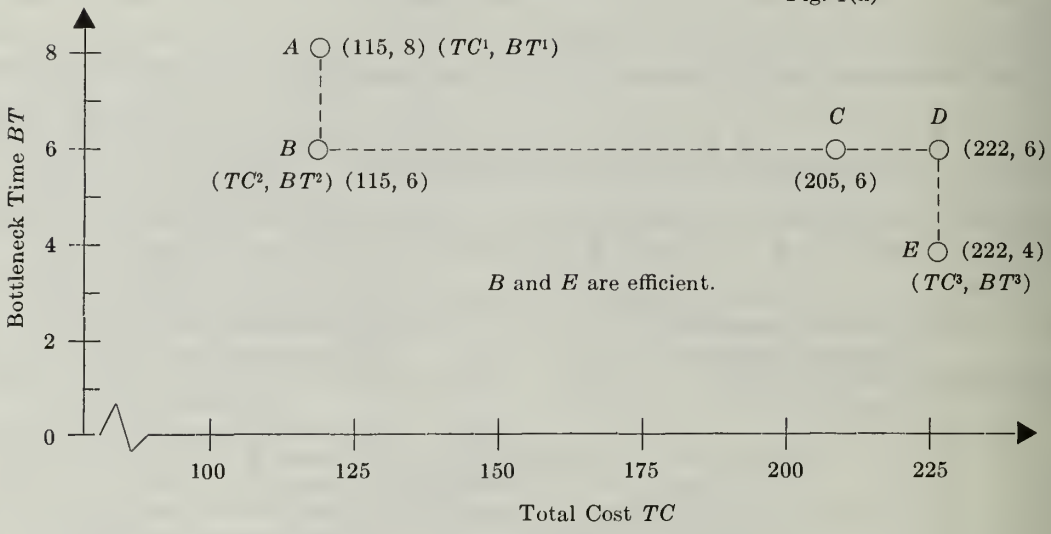


Fig. 1(i) Efficient pairs (TC, BT)

Figure 1 (contd.). Computing efficient pairs (TC, BT)

4 AN ALGORITHM FOR THE BOTTLENECK TRANSPORTATION PROBLEM

The Algorithm of Section 3 determines the set of efficient solution pairs $\{(TC^1, BT^1), (TC^2, BT^2), \ldots, (TC^k, BT^k)\}$. Suppose $c_{ij}=0$ for $i\epsilon I$ and $j\epsilon J$. Then $TC^1=TC^2=\ldots=TC^k=0$ so that from step (10) of Algorithm 3, we find that $l=1$ and the solution with the minimum BT (i.e., the optimal solution for the BTP) is the only efficient solution. (More generally, if $\{c_{ij}\}$ can be expressed as $\{\xi+\alpha_i+\beta_j\}$ then the total cost TC is independent of the solution so that the only efficient solution corresponds to minimum BT). As may be expected, Algorithm 3 becomes considerably simpler for this special case (because the dual to the cost minimization problem is not needed) and is presented as Algorithm 4 below (steps are numbered to correspond to Algorithm 3). Later we will discuss the relationship of Algorithm 4 with the algorithms of Szwarc [15, 16] and Hammer [7].

ALGORITHM 4: For obtaining the minimum BT for a transportation problem:

Step (1) Set $k=1$, let T be the given matrix of times $\{t_{ij}\}$ and $\{a_i\}$ and $\{b_j\}$ be the warehouse availabilities for $i\epsilon I$ and market requirements for $j\epsilon J$ respectively. Obtain an initial basic feasible solution $X=\{x_{ij}\}$ with basis B by any method [12].

Step (2) Compute BT^k using Equation (5).

Step (3) Choose a $(p, q)\ \epsilon\ B$ for which $x_{pq}>0$ and $t_{pq}=BT^k$. If there is no such cell set $k=k+1$ and go to (2). Otherwise choose the cell (p, q) with the largest x_{pq} (Break ties in any arbitrary way.). Let $l=1$. Let $\{x_{ij,\,1}\}$ denote the current solution $\{x_{ij}\}$ and B_1 denote the current basis B.

Step (4) Determine I_R, J_R, I_C, J_C corresponding to (p, q) and B_l. Set $\psi=[(I_R\times J_C)-\{(p, q)\}]$. If ψ has no cell (e, f) such that $t_{ij}<BT^k$ go to (10). Otherwise choose a cell $(e, f)\ \epsilon\ \psi$ for which $t_{ef}<BT^k$.

Step (6 & 7) Same as steps (6)–(7) of Algorithm 3. (There is no step (5)).

Step (8) Set $B_{l+1}=B_l+\{(e, f)\}-\{(r, s)\}$. If $(p, q)=(r, s)$ set $B=B_{l+1}$ and go to (3). Otherwise set $l=l+1$ and go to (4).

Step (9) Set $X_{l+1}=X_l$, $B_{l+1}=B_l+\{(e, f)\}-\{(r, s)\}$, $l=l+1$ and go to (4).

Step (10) The current solution $X=X_l$ is optimal with bottleneck time BT^k. Stop.

JUSTIFICATION: Algorithm 4 is the same as Algorithm 3 except that the costs $\{c_{ij}\}$, TC^k are all zero and hence need not be considered. Consequently no dual variables $\{u_i\}$, $\{v_j\}$ are needed. In step (1), since all the costs are zero, any basic feasible solution is also basic optimal. In step (4), the condition $c_{ij}^{k+1}\neq M$ is replaced by $t_{ef}<BT^k$. This is equivalent since in Algorithm 3 $c_{ij}^{k+1}=M\langle\Rightarrow\rangle (i, j)\notin B$ and $t_{ij}\geq BT^k$ (see step (2) of Algorithm 3). But from Remark 3, $(i, j)\epsilon[I_R\times J_C]-\{(p, q)\}\langle\Rightarrow\rangle (i, j)\notin B$. Consequently $c_{ij}^{k+1}\neq M\langle\Rightarrow\rangle t_{ij}<BT^k$ for $(i, j)\epsilon\psi$. Thus by Theorem 6, Algorithm 4 also determines the set of all efficient solutions (in this case only the minimum BT) in a finite number of steps.

The algorithms by Szwarc [15, 16] and Hammer [7] for the BTP are equivalent [16, p. 474] in that they produce the same sequence of solutions X leading to the optimum BT provided we start from the same initial basic feasible solution. Algorithm 4 can be verified to be identical to Szwarc's algorithm [16, p. 478] provided we choose a cell (e, f) with minimum t_{ef} in Ψ in step (4) of Algorithm 4.

Algorithm 4 has been coded in FORTRAN V and run on the UNIVAC 1108 computer. For this purpose, we needed to specify (i) a method for finding an initial basic feasible solution in step (1) of Algorithm 1 and (ii) a rule for selecting a cell $(e, f)\ \epsilon\ \psi$ with $t_{ef}<BT^k$ in step (4) of Algorithm 4.

Several methods and rules can be suggested for (i) and (ii), and computational experiments will be needed to determine the "best" (in terms of overall solution time). An alternate approach would be to take advantage of the similarity between the *BTP* and *STP* and use the results in [12]. In [12] it is shown that for the *STP* with $m \leq n$, a "Modified Column Minimum Rule" performed best as the method for finding an initial basic feasible solution. This (with $\{t_{ij}\}$ replacing $\{c_{ij}\}$ in [12]) was coded as the method in step (1) of Algorithm 4. Again for *STP* with $m \leq n$, the "Row Most Negative Rule" performed best in the choice of a pivot. A modification of this rule was used in step (4) of Algorithm 4 for the choice of (e, f). For instance, assuming $|I_R| \leq |J_C|$ (where $|\ |$ denotes the cardinality of (number of elements in) a set), we first examine the cells along the first row index say i_1 of I_R. Let

$$t_{i_1 j^*} = \underset{j \epsilon J_C}{\text{Min}}\ t_{i_1 j}.$$

If $t_{i_1 j^*} < t_{p_1 q}$ then we choose (i_1, j^*) to be the entering cell (e, f). If, however, $t_{i_1 j^*} \geq t_{pq}$, we proceed to the second index i_2 of I_R, etc. (If $|I_R| > |J_C|$ then the search would start with the first column index in J_C etc.). It should be noted that this rule is quite different from the one in [7] and [15, 16] where t_{ef} is chosen as the smallest t_{ij} in ψ (similar to the "Matrix Most Negative Rule" for the *STP* which performs rather poorly compared to the "Row Most Negative Rule" [12, p. 201] in terms of overall computation time).

Mean computational times for Algorithm 4 for several (100×100) (i.e., $m=n=100$) randomly generated *BTP* are presented in Table 1. The solution times are relatively insensitive to the ranges for the parameters $\{a_i\}$ and $\{b_j\}$. However, the solution times more than double when the range for $\{t_{ij}\}$ is increased from 0–9 to 0–99, but are insensitive to larger ranges for $\{t_{ij}\}$. The solution times are also quite insensitive to the probability of infeasibility θ (each cell $(i, j) \epsilon (I \times J)$ is declared infeasible (i.e., $x_{ij}=0$) in *any* solution with a probability of θ). Thus the solution procedure is quite robust (insensitive to parameter changes) and computationally efficient (100×100 transportation or assignment problems take only about a second on the average). (For 185×185 problems these times increase to about 3.4 seconds on the average so that the solution times goes up approximately as n^2 for $(n \times n)$ problems.) These times are smaller than the corresponding times for solving standard transportation problems (about 2 seconds on the UNIVAC 1108 for 100×100 standard transportation problems (a_i, b_j in the range 0–99; c_{ij} in the range 0–99) and assignment problems (c_{ij} in the range 0–99) [12]).

TABLE 1. *Computational Performance of Algorithm 4 for the BTP*

TABLE 1(a). *Mean Solution Times (sec.) for (100×100) Transportation Problems ($\theta=0$)*

$\{a_i\}$, $\{b_i\}$ $\{t_{ij}\}$	0–9	0–99	0–999	0–9999
0–9	0. 456	1. 088	0. 936	0. 993
0–99	0. 490	1. 229	1. 123	1. 023
0–999	0. 521	1. 168	1. 112	1. 091

TABLE 1(b). *Mean Solution Times (sec.) for (100×100) Assignment Problems ($\theta=0$)*

Assignment	0. 490	1. 065	1. 054	1. 100

TABLE 1(c). *Mean Solution Times (sec.) for (100×100) Problems as a Function of θ ($\{t_{ij}\}$ are in the range 0–99)*

Probability of Infeasibility θ	0	0. 2	0. 4	0. 6	0. 8
Transportation $\{a_i\}$, $\{b_j\}$ 0–99	1. 229	0. 985	1. 371	1. 133	1. 218
Assignment	1. 065	0. 975	0. 998	0. 944	0. 954

NOTES: (1) All computational times are mean times in seconds on the UNIVAC 1108 (FORTRAN V) based on 21 randomly generated problems each.

(2) The parameters $\{a_i\}$, $\{b_j\}$ and $\{t_{ij}\}$ are randomly chosen integers distributed uniformly in the ranges specified.

(3) θ= Probability of infeasibility, i.e., the probability that a cell is declared inadmissible in any solution

A comparison of these computational times with those of the threshold algorithm of Garfinkel and Rao [5] indicates that Algorithm 4 and hence the Szwarc-Hammer algorithm is computationally more efficient. For 100×100 transportation problems with $\{a_i\}$, $\{b_j\}$ in the range 0–100 the computational times reported for the threshold algorithm are 4.1, 7.4 and 3.4 seconds (about 5 seconds, on the average) on the IBM 360/65 [5] (the ranges for $\{t_{ij}\}$ for these three problems were 0–100, 0–500 and 0–1000 respectively). By comparison, Algorithm 4 takes only about 1.12–1.23 seconds on the UNIVAC 1108. Even after adjusting for the difference between the computers (the IBM 360/65 takes about 50% more time than the UNIVAC 1108) these results still show that Algorithm 4 takes only about 35% of the time required for the threshold algorithm.

5. DETERMINATION OF EFFICIENT PARIS (TC, BT) AND (TC, SB)

In this section we provide an algorithm which in addition to determining the efficient pairs (C, BT) also provides efficient pairs (TC, SB) for each value of BT. This is accomplished by Algorithm 5 below which is again a variant of Algorithm 1.

ALGORITHM 5. For obtaining all efficient pairs (TC, BT) and for each such BT all efficient pairs (TC, SB).

This algorithm is identical to Algorithm 1 except that step (2) is replaced by step (2″) below and there is a step (5″) below following step (4) of Algorithm 1.

Step (2″) (a) Set $\Pi=\{(i, j) \in [I\times J] | t_{ij} > BT^k\}$

and

$$\Omega=\{(i, j) \in [I\times J] | t_{ij}=BT^k\}.$$

(b) Drive the amount

$$SB=\sum_{(i,j)\in\Omega} x_{ij}$$

down to zero while maintaining TC-optimality among Π-solutions. (The method for reducing SB to zero will be provided later).

Step (2'') (c) If SB can not be driven to zero, go to step (4). Otherwise go to step (3).

Step (5'') The list of efficient solution pairs (TC, SB) corresponding to each BT^j $(j=1,2, \ldots, l)$ were those determined while minimizing SB (i.e., the sum of x_{ij} over the set Ω defined for this value of BT^j) after eliminating (if any) $(\hat{TC}, \hat{SB})$ such that there exists a (TC', SB') with $TC'=\hat{TC}$ and $SB'<\hat{SB}$.

JUSTIFICATION: The set Π defined in step 2(a) of Algorithm 1 is the same as the union of the sets Π and Ω defined in step 2'' (a) of Algorithm 5. Consequently step 2(b) of Algorithm 1 and step 2'' (c) of Algorithm 5 are the same and hence the algorithms are equivalent in obtaining efficient pairs (TC^j, BT^j). Furthermore, by a proof very similar to that for Theorem 1, it can be proved that for each such BT^j the set of all efficient-pairs $\{(TC, SB)\}$ are obtained by Algorithm 5.

We now consider an operational procedure for driving the amount SB down while maintaining TC-optimality among Π-solutions. By a logic very similar to that advanced in Section 3, it is easily seen that the set of all efficient pairs (TC, SB) for a given value of BT^k can be obtained by solving the compound objective

$$H=F_1+\delta F_2=\sum_{i\epsilon I}\sum_{j\epsilon J} c_{ij}x_{ij}+\delta \sum_{(i,j)\epsilon\Omega} x_{ij} \tag{33}$$

where

$$F_1=\sum_{i\epsilon I}\sum_{j\epsilon J} c_{ij}x_{ij}=TC, \tag{34}$$

and

$$F_2=\sum_{(i,j)\epsilon\Omega} x_{ij}=SB. \tag{35}$$

The problem of minimizing (33) among Π-solutions can be formulated as a standard transportation problem P^A below by suitably defining the cost coefficients $C^A=\{c_{ij}{}^A\}$ as follows:

$$c_{ij}{}^A=\begin{cases}\infty & \text{for } (i,j)\epsilon\Pi' \\ c_{ij}+\delta & \text{for } (i,j)\epsilon\Omega \text{ (i.e., } t_{ij}=BT^k) \\ c_{ij} & \text{elsewhere, where}\end{cases} \tag{36}$$

$$\Pi'=\{(i,j)|t_{ij}>BT^k \text{ and } (i,j)\notin B\}. \tag{37}$$

Since $c_{ij}{}^A=\infty$ for $(i,j)\epsilon\Pi'$, the objective of minimizing the total cost with coefficients $\{c_{ij}{}^A\}$ will maintain $x_{ij}=0$ for $(i,j)\epsilon\Pi'$. But our objective is to maintain a Π-solution where $\Pi=\{(i,j)$ such that $t_{ij}>BT^k\}$. Consequently $\Pi'\subseteq\Pi$ and the elements of $(\Pi-\Pi')$ are $(i,j)\epsilon B$ such that $t_{ij}>BT^k$. By the definition of BT^k it follows that $x_{ij}=0$ for $(i,j)\epsilon\Pi-\Pi'$. As will be seen later, x_{ij} will be maintained at zero for these cells as well, which is, after all, the only requirement of a Π-solution. Consequently the objective $\sum_{i\epsilon I}\sum_{j\epsilon J} c_{ij}{}^A x_{ij}$ is precisely the same as H (Equation (33)) for Π-solutions.

Let us now assume that we have the basic optimal solution $\{x_{ij}\}$, $\{u_i\}$ and $\{v_j\}$ to the problem P, i.e., problem P^A when $\delta=0$. The optimal solutions to P^A as δ is parametrically increased from 0 to ∞ can be obtained from the optimal solution to P via the *Area Cost Operator* $\delta\mathbf{C}_A$ discussed in [10, 11]. We present below a simplified version of the results [10, 11], and hence the reader need not refer to those references. The Area Cost Operator $\delta\mathbf{C}_A$ transforms the optimal solutions $\{x_{ij}\}$, $\{u_i\}$

$\{v_j\}$ of P to the optimal solutions $\{x_{ij}^A\}$, $\{u_i^A\}$, $\{v_j^A\}$ for the problem P^A whose data (i.e., , $\{a_i\}$, $\{b_j\}$) are the same as those of P except $c_{ij}^A = c_{ij} + \delta\gamma_{ij}$ for $i \epsilon I$ and $j \epsilon J$ ($\delta \geq 0$). Defining

$$\gamma_{ij} = \begin{cases} 1 & \text{for } (i, j) \epsilon \Omega \\ 0 & \text{elsewhere} \end{cases}$$

readily verified that this is consistent with the definition of c_{ij}^A in (36). (Note however that now denote the unit costs for $\delta = 0$. Thus from (36) $c_{ij} = \infty$ for $(i, j) \epsilon \Pi'$ and c_{ij} elsewhere.) operator is said to be *basis preserving* and denoted by light face letters δC_A if the optimal s B^A to P^A is same as the optimal basis B to P. Let μ be the maximum extent to which the basis erving operator is valid and let H and H^A denote the optimal values of the objective function P and P^A.

THEOREM 7 (in part, adapted from Theorems 5(c) and 8 of [10]): For δC_A, the primal solu- is given by

$$x_{ij}^A = x_{ij} \qquad \text{for } i \epsilon I \text{ and } j \epsilon J.$$

dual solutions $\{u_i^A\}$ and $\{v_i^A\}$ are given by

$$u_i^A = u_i + \delta u_i' \qquad \text{for } i \epsilon I$$

$$v_j^A = v_j + \delta v_j' \qquad \text{for } j \epsilon J$$

e

for $i \epsilon I$ and $\{v_j'\}$ for $j \epsilon J$ are solutions to the "triangular" set of equations (easily solved similar termining the duals u_i and v_j from c_{ij} for a basis (see Equation (8)):

$$u_i' + v_j' = \gamma_{ij} \qquad \text{for } (i, j) \epsilon B.$$

hermore, H^A is given by $H + \delta \sum_{(i, j) \epsilon \Omega} x_{ij}$ and the maximum extent μ is given by

$$\mu = \operatorname*{Min}_{(i, j) \epsilon \psi} [(c_{ij} - u_i - v_j)/(u'_i + v'_j - \gamma_{ij})]$$

e

$$\psi = \{(i, j) \epsilon [I \times J] | (u'_i + v'_j - \gamma_{ij}) > 0 \text{ and } (i, j) \notin \Pi'\}.$$

$e, f) \epsilon \psi$ be the cell at which the minimum of (42) is attained. If, however, the set ψ is empty, e the current value of $\sum_{(i, j) \epsilon \Omega} x_{ij}$ is the lowest attainable value for $F_2 = SB$.

PROOF: Since the operator is basis preserving $B^A = B$. Since the $\{a_i\}$ and $\{b_j\}$ have not ged, Equation (38) holds. From (33) it then readily follows that $H^A = H + \delta \sum_{(i, j) \epsilon \Omega} x_{ij}$.

We now show that $\{u_i^A\}$, $\{v_j^A\}$ satisfy (8)–(9) for P^A. For $(i, j) \epsilon B^A = B$, from (39)–(41) and from acts that $c_{ij}^A = c_{ij} + \delta\gamma_{ij}$ and $c_{ij} = u_i + v_j$, it follows that $c_{ij}^A = c_{ij} + \delta\gamma_{ij} = (u_i + v_j) + \delta(u'_i + v'_j) = v_j^A$ so that (8) is satisfied. Furthermore, Equation (9) implies $c_{ij}^A = c_{ij} + \delta\gamma_{ij} \geq u_i^A + v_j^A = v_j + \delta(u'_i + v'_j)$ so that $(c_{ij} - u_i - v_j) \geq \delta(u'_i + v'_j - \gamma_{ij})$, and this is satisfied by the choice of μ

and ψ in Equations (42)–(43) (recall that $\delta \leq \mu$ and the conditions (9) are always satisfied fo $(i, j) \epsilon \Pi'$ since $c_{ij} = \infty$ for these cells).

Now suppose the set $\psi = \phi$. Then by the preceding arguments it follows that the solution (38)–(41) define an optimal solution for P^A for *all* $\delta \geq 0$. Consequently, as $\delta \to \infty$, this solution i still optimal and hence by Theorem 3, the minimum value for $SB = \sum_{(i,j)\epsilon\Omega} x_{ij}$ has been attained.

Q.E.D

For applying the operator δC_A for $\delta > \mu$, a sequence of basis preserving δC_A can be applied The sequence $[B_l, X_l, \{u_{t,\ l}\}, \{v_{j,\ l}\}]$ can be determined in a manner similar to those discussed i Section 3 for applying δC^+_{pq} with $\delta > \mu$. Theorem 8 below is analogous to Theorem 5 of Section

THEOREM 8: The solutions X_l and X_{l+1} are such that

$$SB_{l+1} = SB_l - (u'_e + v'_f - \gamma_{ef})\Delta. \tag{44}$$

PROOF: By the well known [1, p. 531] relationship between the steppingstone method [and the u-v or MODI method [3],

$$\gamma_{ef} - u'_e - v'_f = \sum_{(i,j)\epsilon\Gamma_2} \gamma_{ij} - \sum_{(i,j)\epsilon\Gamma_1} \gamma_{ij}$$

where Γ_1 and Γ_2 are the set of odd parity ("givers") and even parity ("getters") cells in the cycle created by the addition of (e, f) to B. From Equation (27).

$$SB_{l+1} - SB_l = \sum_{\Omega} (x_{ij,\ l+1} - x_{ij,\ l}) = -\Delta[|\Gamma_1 \cap \Omega| - |\Gamma_2 \cap \Omega|]$$

where $||$ denotes the cardinality of the set. Since $\gamma_{ij} = 1$ for $(i, j) \epsilon \Omega$ and 0 otherwise, it follows th

$$SB_{l+1} - SB_l = -\Delta\left[\sum_{(i,j)\epsilon\Gamma_1} \gamma_{ij} - \sum_{(i,j)\epsilon\Gamma_2} \gamma_{ij}\right] = -\Delta(u'_e + v'_f - \gamma_{ef}).$$

Q.E.I

Since the transformation from X_l to X_{l+1} was described as an alternate optimum, it follo that $H_l = H_{l+1}$, i.e., $F_{1,\ l} + \delta F_{2,\ l} = F_{1,\ l+1} + \delta F_{2,\ l+1}$ so that $F_{1,l+1} = F_{1,l} + \delta(F_{2,l} - F_{2,\ l+1})$. From Theore 8 and Equation (34)–(35), it then follows that

$$TC_{l+1} = TC_l + \delta\Delta(u_e' + v'_f - \gamma_{ef}). \tag{45}$$

The maintenance of a Π-solution for $(i, j) \epsilon \Pi'$ was accomplished by declaring $c_{ij} = \infty$ for $(i, j) \epsilon$ I We also have to maintain $x_{ij} = 0$ for $(i, j) \epsilon \Pi\text{-}\Pi' = \{(i, j) \epsilon B | t_{ij} > BT^k\}$. This will be done by the sa procedure adopted in Section 3, i.e., when the cycle Γ includes a $(\Pi\text{-}\Pi')$ cell (r, s) in Γ_2, then we s $X_{l+1} = X_l$ and $B_{l+1} = B_l + \{(e, f)\} - \{(r, s)\}$.

Algorithm 6 below gives a detailed procedure for obtaining all efficient pairs (TC, BT) a for each such BT, all the efficient pairs (TC, SB). It is the same as Algorithm 5 together with t operationalization discussed earlier for driving SB down to zero while maintaining TC-optimali among Π-solutions.

ALGORITHM 6: For obtaining all efficient pairs (TC, BT) and for each such BT, all the e cient pairs (TC, SB). (The steps are numbered to correspond to Algorithm 3.)

Step (1) Same as step (1) of Algorithm 3.

Step (2) Initialize $TC^{k+1}=TC^k$ and compute BT^k using (5).

Derive the cost matrix C^{k+1} from C^k as follows:

$$c_{ij}^{k+1}=\begin{cases}M \text{ if } t_{ij}>BT^k \text{ and } (i,j)\notin B\\ c_{ij}{}^k \text{ otherwise}\end{cases}$$

where M is a very large positive number (e.g. 10^{30}).

Step (3) Define $\Omega=\{(i,j)\epsilon[I\times J]|t_{ij}=BT^k\}$. Let $\delta=0$, $l=1$. Denote by $\{x_{ij,1}\},\{u_{i,1}\}$ and $\{v_{j,1}\}$ the current values of $\{x_{ij}\}$, $\{u_i\}$ and $\{v_j\}$. Let B_1 denote the current basis. Determine $SB_1=\sum_{(i,j)\epsilon\Omega} x_{ij,1}$. Set

$$\gamma_{ij}{}^k=\begin{cases}1 \text{ if } (i,j)\epsilon\Omega\\ 0 \text{ otherwise.}\end{cases}$$

Step (4) Solve for $\{u'_i\}$ and $\{v'_j\}$ satisfying $u'_i+v'_j=\gamma_{ij}^k$ for $(i,j)\epsilon B_l$. Let $\psi=\{(i,j)|u'_i+v'_j-\gamma>0$ and $c_{ij}^k\neq M\}$. If ψ is empty, go to step (10). Otherwise compute

$$\mu_l=\min_{(i,j)\epsilon\psi}[c_{ij}^{k+1}-u_{i,l}-v_{j,l})/(u'_i+v'_j-\gamma_{ij}^k)]$$

and record the cell (e,f) at which the minimum is attained.

Step (5) Set $u_{i,\,l+1}=u_{i,\,l}+\mu_l u'_i$ for $i\epsilon I$ and $v_{j,\,l+1}=v_{j,\,l}+\mu_l v'_j$ for $j\epsilon J$.

Step (6) Redefine $c_{ij}^{k+1}=c_{ij}^{k+1}+\mu_l$ for $(i,j)\epsilon\Omega$. Set $\delta=\delta+\mu_l$. Determine the cycle Γ created by the addition of (e,f) to B_l. If there is an $(r,s)\epsilon\Gamma_2$ with $x_{rs,\,l}=0$ and $t_{rs}>BT^k$ go to step (9). Otherwise go to step (7).

Step (7) Let $\chi=\{(g,h)\}$ be the set of cells such that $x_{gh,\,l}$ is the minimum of $x_{ij,\,l}$ for $(i,j)\epsilon\Gamma_1$. *Randomly* choose $(r,s)\epsilon\chi$ (i.e., each element in χ has equal probability of being chosen as (r,s)). Let $\Delta=x_{rs,\,l}$. Set

$$x_{ij,\,l+1}=\begin{cases}x_{ij,\,l}-\Delta & \text{for } (i,j)\epsilon\Gamma_1\\ x_{ij,\,l}+\Delta & \text{for } (i,j)\epsilon\Gamma_2\\ x_{ij,\,l} & \text{elsewhere.}\end{cases}$$

Step (8) If $t_{rs}>BT^k$ redefine $c_{rs}^{k+1}=M$. Set $TC^{k+1}=TC^{k+1}+\delta\Delta(u'_e+v'_f-\gamma_{ef}{}^k)$. Set $B_{l+1}=B_l+\{(e,f)\}-\{(r,s)\}$. Set $SB_{l+1}=SB_l-\Delta(u'_e+v_f'-\gamma_{ef}^k)$ and record the current (TC^{k+1}, SB_{l+1}) as a potentially efficient solution. If $SB_{l+1}=0$ set $k=k+1$, $B=B_{l+1}$ and go to step (2). Otherwise set $l=l+1$ and go to step (4).

Step (9) Set $X_{l+1}=X_l$, $SB_{l+1}=SB_l$. Set $c_{rs}^{k+1}=M$, $B_{l+1}=B_l+\{e,f)\}-\{(r,s)\}$, $l=l+1$ and go to (4).

Step (10) Same as step (10) of Algorithm 3.

Step (11) Same as step (5″) of Algorithm 5.

THEOREM 9: Algorithm 6 determines the set of all efficient pairs (TC, BT), and for each such BT the set of efficient pairs (TC, SB) with "probability one".

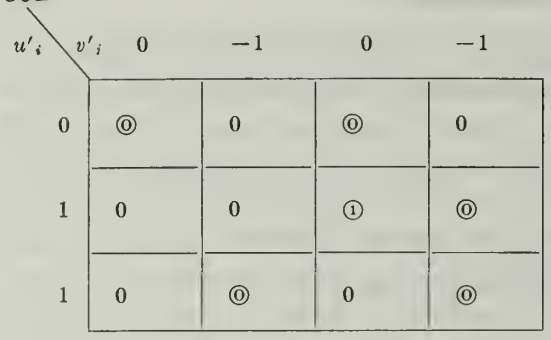
PROOF: Algorithm 6 differs from Algorithm 5 only in the way step 2''(b) is executed. Theorems 7–8 and the related exposition show that Algorithm 6 drives *SB* down while maintaining *TC*-optimality among Π-solutions. That the *area cost operator* $\delta \mathbf{C}_A$ can be applied in a finite number of steps for *nondegenerate problems* follows from (44) since for such problems $\Delta > 0$, (in fact, $\Delta \geq 1$ if the $\{a_i\}$, $\{b_j\}$ are integers) $\gamma_{ef} - u'_e - v'_f \geq 1$ (this again follows from the fact that $\{\gamma_{ij}\}$ and hence the $\{u'_i\}$ and $\{v'_j\}$ are integers) and since *SB* is bounded below by zero. For degenerate problems however, the amount Δ can be zero and hence a basis may be repeated and the algorithm may cycle (circle). However, by the random choice rule of step (7) the algorithm would converge with "probability one" (similar to the random choice rule for the simplex method of linear programming [3, p. 123]. The standard perturbation procedures [1, 3] to guarantee nondegeneracy (to avoid cycling) are not readily applicable here since an $x_{ij} = \epsilon$ (a small positive number) is quite different from an $x_{ij} = 0$ as far as the objective *BT* is concerned. (The perturbed problem's optimum *BT* can turn out to be larger than the optimum *BT* for the unperturbed problem). However, "cycling" is no "real" threat in that it seldom occurs in practice. The possibility of "cycling" for Algorithm 6 is as remote as that for the *u-v* or MODI method [3] for the *STP*. The authors have solved over 1000 assignment problems (very degenerate) of size over $m = n = 100$ by the MODI method and cycling has never occurred. In any event, the random choice rule assures us of an optimum solution with "probability one".

EXAMPLE: Figure 2 shows the calculation of (*TC*, *BT*) and (*TC*, *SB*) for the same problem shown in Figs. 1 (a)–(b). We now indicate step by step the computations:

Step	Result
(1)	$k=1$, $TC^1=115$. Shown in Figure 1(a).
(2)	$TC^2=115$, $BT^1=8$. C^2 shown in Figure 1(c).
(3)	$\Omega=\{(2, 3)\}$, $\delta=0$, $l=1$, $SB_1=8$, $\{\gamma_{ij}^1\}$ shown in Figure 2(a). X_1, $\{u_{i,1}\}$, $\{v_{i,1}\}$ shown in Figure 1(c).
(4)	$\{u'_i\}$, $\{v'_j\}$ shown in Figure 2(a). $\psi=\{(2, 1), (3, 1), (3, 3)\}$. $\mu_1=0$, $(e, f)=(3, 3)$.
(5)	$\{u_{i,2}\}$, $\{v_{j,2}\}$ shown in Figure 1(d), $c_{23}^2=1$, $\delta=0$.
(6)	Cycle Γ shown in Figure 1(c).
(7)	$\chi=\{(2, 3), (3, 4)\}$. $(r, s)=(2, 3)$. $\Delta=8$. X_2 shown in Figure 1(d).
(8)	$TC^2=115$. $B_2=B_1+\{(3, 3)\}-\{(2, 3)\}$. $SB_2=0$. $k=2$, $B=B_2$.
(2)	$TC^3=115$, $BT^2=6$, C^3 shown in Figure 2(b).
(3)	$\Omega=\{(1, 2), (1, 3), (3, 2)\}$. $\delta=0$, $l=1$. $SB_1=19$. $\{\gamma_{ij}^2\}$ shown in Figure 2(c).
(4)	$\{u'_i\}$, $\{v'_j\}$ shown in Figure 2(c). $\psi=\{(1, 2), (1, 4), (2, 2)\}$. $\mu_1=4$, $(e, f)=(1, 2)$.
(5)	$\{u_{i,2}\}$, $\{v_{j,2}\}$ shown in Figure 2(d). $c_{12}^3=11$, $c_{13}^3=7$, $c_{32}^3=8$, $\delta=4$.
(6)	Cycle Γ shown in Figure 2(b).
(7)	$(r, s)=(1, 3)$. $\Delta=9$. X_2 shown in Figure 2(d).
(8)	$TC^3=115+(4\times 9\times 1)=151$. $SB_2=19-9=10$. $l=2$.
(4)	$\{u'_i\}$, $\{v'_j\}$ shown in Figure 2(e). $\psi=(e, f)=(2, 2)$, $\mu_2=1$.
(5)	$\{u_{i,3}\}$, $\{v_{j,3}\}$ shown in Figure 2(f). $c_{12}^3=12$, $c_{13}^3=8$, $c_{32}^3=9$, $\delta=5$.
(6)	Cycle Γ shown in Figure 2(d). $(r, s)=(3, 4)\epsilon\Gamma_2$.
(9)	$X_3=X_2$ shown in Figure 2(f). $SB_3=10$, $c_{34}^3=M$. $B_3=B_2+\{(2, 2)\}-\{(3, 4)\}$. $l=3$.
(4)	$\{u'_i\}$, $\{v'_j\}$ shown in Figure 2(g). $\psi=(e, f)=\{(1, 4)\}$, $\mu_3=1$.
(5)	$\{u_{i,4}\}$, $\{v_{j,4}\}$ shown in Figure 2(h). $c_{12}^3=13$, $c_{13}^3=9$, $c_{32}^3=10$, $\delta=6$.
(6)	Cycle Γ shown in Figure 2(f).
(7)	$\chi=(r, s)=(1, 2)$. $\Delta=9$, X_4 shown in Figure 2(h).
(8)	$TC^3=151+(6\times 9\times 1)=205$. $B_4=B_3+\{(1, 4)\}-\{(1, 2)\}$. $SB_4=10-9=1$. $l=4$.
(4)	$\{u'_i\}$, $\{v'_j\}$ shown in Figure 2(i). $\psi=(e, f)=\{(3, 1)\}$, $\mu_4=11$.
(5)	$\{u_{i,5}\}$, $\{v_{j,5}\}$ shown in Figure 2(j). $c_{12}^3=24$, $c_{13}^3=20$, $c_{32}^3=21$, $\delta=17$.

ep	Result

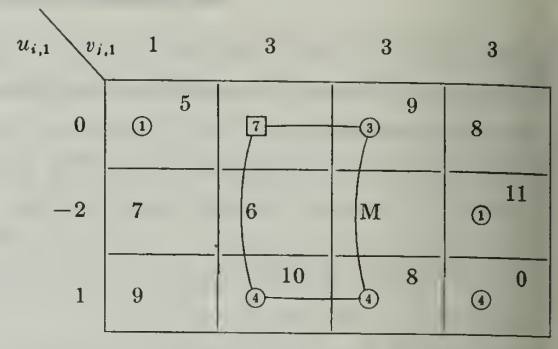
5) Γ shown in Figure 2(h).
7) $\chi=(r, s)=(3, 2)$. $\Delta=1$, X_5 shown in Figure 2(j).
3) $TC^3=205+(17\times1\times1)=222$. $B_5=B_1+\{(3, 1)\}-\{(3, 2)\}$. $SB_1=1-1=0$. $k=3$, $B=B_5$.
2) $TC^4=222$. $BT^3=4$. C^4 shown in Figure 2(k).
3) $\Omega=\{(1, 1), (1, 4), (2, 4), (3, 3)\}$. $\delta=0$, $l=1$, $SB_1=32$. $\{\gamma_{i,}{}^3\}$ shown in Figure 2(l).
4) $\{u'_i\},\{v'_j\}$ shown in Figure 2(l). $\psi=(e, f)=(2, 1)$, $\mu_1=13$.
5) $\{u_{i,\ 2}\}$, $\{v_{j,\ 2}\}$ shown in Figure 2(m). $c_{11}{}^4=14$, $c_{14}{}^4=21$, $c_{24}{}^4=14$, $c_{33}{}^4=17$. $\delta=13$.
6) Cycle Γ shown in Figure 2(k).
7) $\chi=(r, s)=(2, 4)$. $\Delta=1$, X_2 shown in Figure 2(m).
8) $TC^4=222+(13\times1\times1)=235$, $B_2=B_1+\{(2, 1)\}-\{(2, 4)\}$. $SB_2=32-1=31$, $l=2$.
4) $\{u'_i\},\{v'_j\}$ shown in Figure 2(n). $\psi=\phi$.
0) Potential efficient solution pairs (TC, BT) are $\{(115, 8), (115, 6), (222, 4)\}$. Thus the only efficient solutions are $\{(115, 6), (222, 4)\}$ shown as B and F in Figure 2(o).
1) For $BT^1=6$, the efficient pairs (TC, SB) are $\{(115, 19), (151, 10), (205, 1), (222, 0)\}$ i.e., points B, C, D, E in Figure 2(o). For $BT^2=4$, the efficient pairs (TC, SB) are $\{(222, 32), (235, 31)\}$, i.e., the points F and G in Figure 2(o).



$k=1$, $l=1$

$[\{\gamma_{ij}^1\}, \{u'_i\}, \{v'_j\}]$

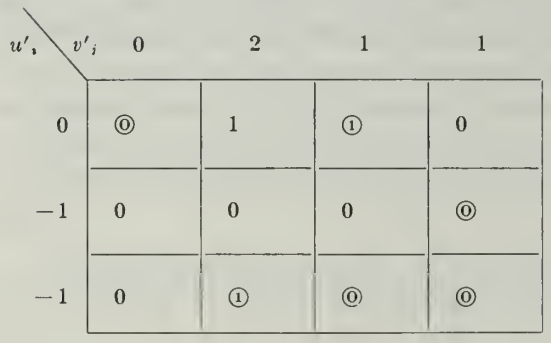
Fig. 2(a)



$k=2$, $TC^3=115$, $BT^2=6$, $SB_1=19$

$l=1$, $\delta=0$, $[C^3, B_1, X_1]$ shown

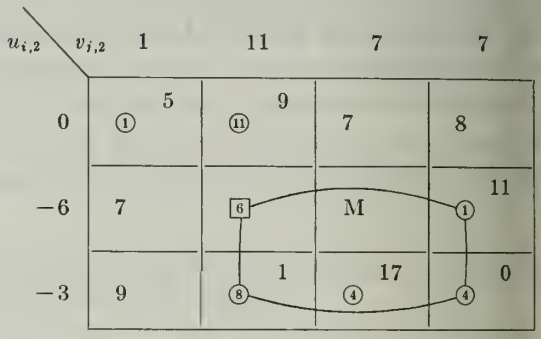
Fig. 2(b)



$k=2$, $l=1$

$[\{\gamma_{ij}^2\}, \{u'_i\}, \{v'_j\}]$

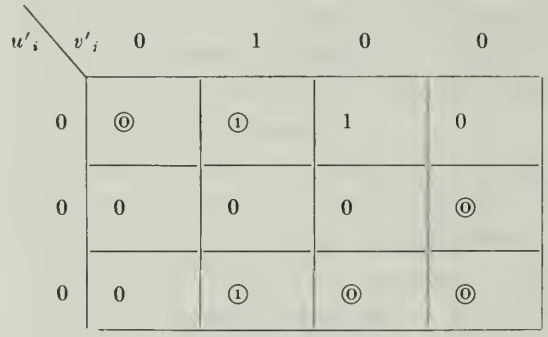
Fig. 2(c)



$k=2$, $TC^3=151$, $SB_2=10$

$l=2$, $\delta=4$, $[C^3, B_2, X_2]$ shown

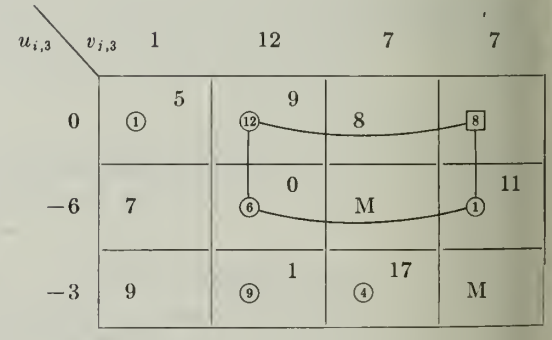
Fig. 2(d)



$k=2$, $l=2$

$[\{\gamma_{ij}^2\}, \{u'_i\}, \{v'_j\}$

Fig. 2(e)

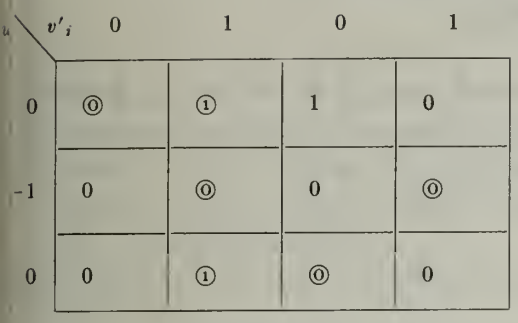


$k=2$, $TC^3=151$, $SB_3=10$

$l=3$, $\delta=5$, $[C^3, B_3, X_3]$ shown

Fig. 2(f)

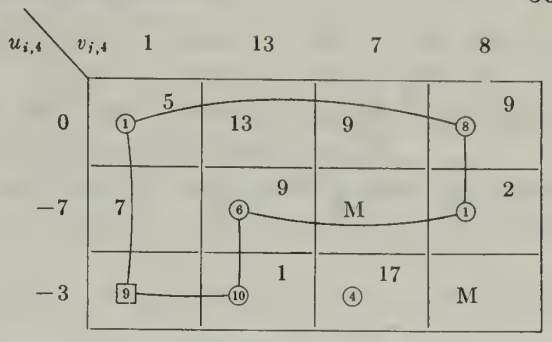
Figure 2. Computing efficient pairs (TC, BT) and (TC, SB)



$k=2$, $l=3$

$[\{\gamma^2_{ij}\}, \{u'_i\}, \{v'_j\}]$

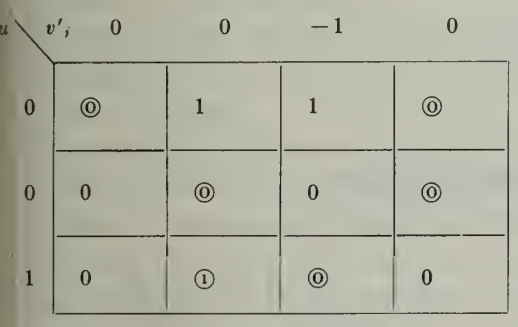
Fig. 2(g)



$k=2$, $TC^3=205$, $SB_4=1$

$l=4$, $\delta=6$, $[C^3, B_4, X_4]$ shown

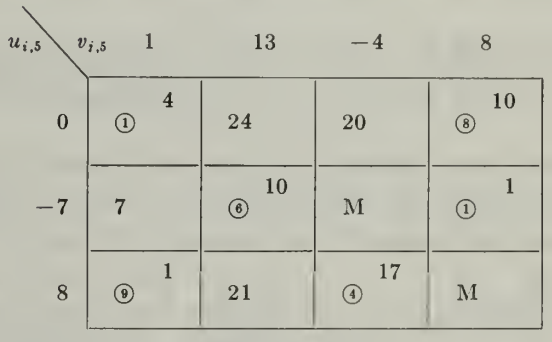
Fig. 2(h)



$k=2$, $l=4$

$[\{\gamma^2_{ij}\}, \{u'_i\}, \{v'_j\}]$

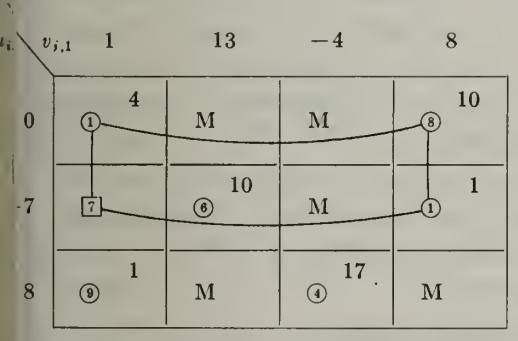
Fig. 2(i)



$k=2$, $TC^3=222$, $SB_5=0$

$l=4$, $\delta=17$, $[C^3, B_5, X_5]$ shown

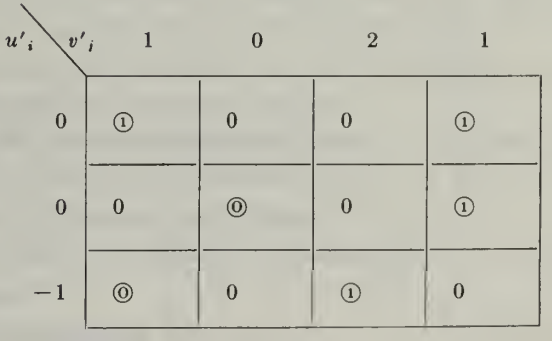
Fig. 2(j)



$k=2$, $TC^4=222$, $BT^3=4$, $SB_1=32$

$l=1$, $\delta=0$, $[C^4, B_1, X_1]$ shown

Fig. 2(k)



$k=3$, $l=1$

$[\{\gamma^3_{ij}\}, \{u'_i\}, \{v'_j\}]$

Fig. 2(l)

Figure 2 (contd.). Computing efficient (TC, BT) and (TC, SB)

$u_{i,2}$ \ $v_{j,2}$	14	13	22	21
0	⑭ 3	M	M	㉑ 11
−7	⑦ 1	⑥ 10	M	14
−5	⑨ 1	M	⑰ 17	M

$k=3$, $TC^4=235$, $SB_2=31$

$l=2$, $\delta=13$, (C^4, B_2, X_2) shown

Fig. 2(m)

u'_i \ v'_j	1	1	2	1
0	①	0	0	①
−1	⓪	⓪	0	1
−1	⓪	0	①	0

$k=3$, $l=2$

$[\{\gamma^3_{ij}\}, \{u'_i\}, \{v'_j\}$

Fig. 2(n)

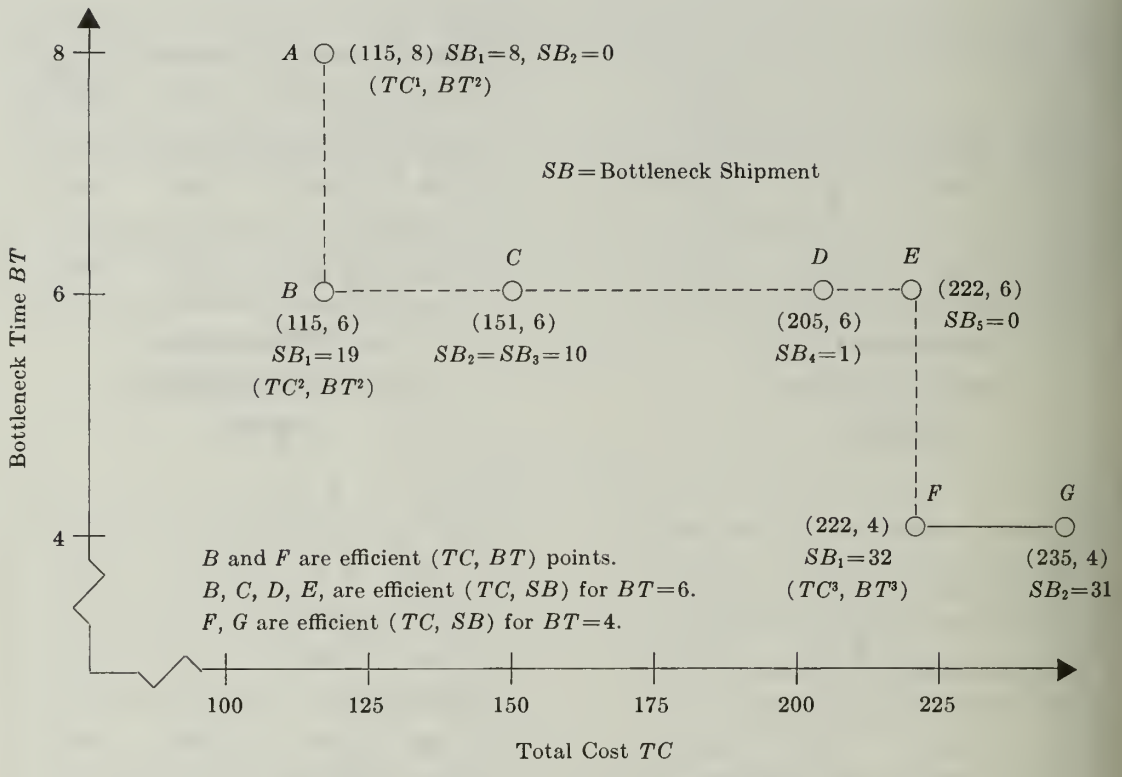


Fig. 2(o) Efficient (TC, BT) and (TC, SB)

Figure 2 (contd.). Computing efficient (TC, BT) and (TC, SB)

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HE CONJUGATE GRADIENT TECHNIQUE FOR CERTAIN QUADRATIC NETWORK PROBLEMS

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ABSTRACT

We consider a class of network flow problems with pure quadratic costs and demonstrate that the conjugate gradient technique is highly effective for large-scale versions. It is shown that finding a saddle point for the Lagrangian of an m constraint, n variable network problem requires only the solution of an unconstrained quadratic programming problem with only m variables. It is demonstrated that the number of iterations for the conjugate gradient algorithm is substantially smaller than the number of variables or constraints in the (primal) network problem. Forty quadratic minimum-cost flow problems of various sizes up to 100 nodes are solved. Solution time for the largest problems (4,950 variables and 99 linear constraints) averaged 4 seconds on the CBC Cyber 70 Model 72 computer.

I TRODUCTION

Network problems with convex costs are frequently used to model transportation systems [4] and scheduling problems [2]. The use of convex cost functions (instead of the much simpler case of linear costs) has been especially important in urban transportation models, where a road network with strictly convex cost functions is used to model congestion of automobile traffic. In these models, the use of a linear network model has led to poor results [1]. Other examples of convex network problems are the stochastic transportation problem and the production scheduling problem with overtime costs [2].

In this paper we address an important class of quadratic convex network problems, where, letting x_{ij} denote the flow along link ij in some network, the cost on the link is $q_{ij}(x_{ij})^2$. (The inclusion of linear terms in the cost functions also can be considered as a straight-forward extension of the results in this paper). Here q_{ij} is a cost function parameter for link ij. Thus we are considering a class of network problems of the form

$$\text{Min} \sum_{ij \in I} q_{ij}(x_{ij})^2 = \text{Min } 1/2x^t Qx$$

Subject to $Ax=b$

where the summation in the objective function is over the n links ij in the network; here I denot the set of links. In (P), x is an n dimensional vector of flows, and Q is an $n \times n$ diagonal matri $2q_{ij}$ is the diagonal entry of Q corresponding to the variable x_{ij} and is not the ij entry of Q. Tl constraints $Ax=b$ in (P) are conservation of flow equations. Now since the objective function (P) is symmetric, there is no reason to impose nonnegativity constraints on the x_{ij}; we simp interpret a negative value of x_{ij} as a flow in the direction from node j to node i.

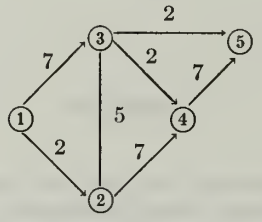

FIGURE 1. A pipeline network with capacities.

Other instances in which problems such as (P) arise are when a link in a network can ha flow in either direction, but not in bouth directions. For example, in the pipeline network sho by Figure 1, it may be possible to pump oil from node 2 to node 3 *or* from node 3 to node 2. If allow flow from node 3 to 2, then the maximum flow from 1 to 5 is 9, while if we allow flow fr 2 to 3, then the maximum flow is only 6. When solving a minimum-cost flow (or maximum flo problem it frequently is not obvious in which direction flow should be allowed on the links. We c model problems such as this by using a free variable for each link; the problem can then be solv by the method shown in the next section.

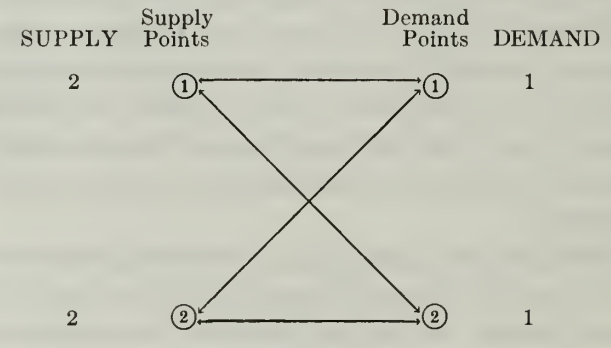

FIGURE 2.

Negative flows are readily interpreted in other contexts as well. For example, in the tra portation problem depicted in Figure 2, suppose we have $x_{11}=2$, $x_{21}=-1$, $x_{12}=0$, $x_{22}=1$. T simply means that two units are shipped from supply point 1 to demand point 1, and one of th units is shipped back to supply point 2, which then sends it to demand point 2. Thus demand poi is actually used as a transshipment point. Note that the total flow into each demand point is Although this particular flow pattern may noy not be optimal, the physical interpretatio plausible.

Problem (P) is a convex programming problem which could be solved by any number of te niques. One possible approach would be to use piecewise linear approximation. Alternatively, th author has successfully implemented the Frank-Wolfe algorithm for solving large-scale convex n work problems more general than (P) [2, 5]. Other possible solution techniques are the gradient p jection technique, the reduced gradient method, and other feasible direction approaches [6]. T special structure of (P), however, is such that it is much simpler to find a saddle point for the L rangian for (P). It is shown in the next section that this requires only the solution of an uncon- st ined quadratic program with fewer variables than (P). Furthermore, the conjugate gradient al rithm [6] exhibits a remarkable efficiency for large-scale problems.

2. METHOD OF SOLUTION

This elimination of nonnegativity requirements from the network problem results in a significant r uction of the computational requirements for solution. Suppose, for example, we have a mini- m m-cost network flow problem with m nodes; let node 1 be the unique source and node m be the d tination. We will let I denote the set of links in the network. Note that we will not have links ij nd ji included in I, since flow from j to i is modeled by a negative value of x_{ij}. Our problem is

$$\text{Min} \sum_{ij \in I} q_{ij}(x_{ij})^2$$

$$\text{s.t.} \sum_{j} x_{1j} = F$$

$$\sum_{i} x_{ij} - \sum_{k} x_{jk} = 0 \qquad j = 2, 3, \ldots, m-1$$

$$-\sum_{i} x_{im} = -F$$

T s is clearly of the form (P). Constraints (2), (3), (4) are conservation of flow requirements at the s rce, each intermediate node, and the destination, respectively. Constraints (2), (3), (4) form an in dence matrix with row rank $m-1$. Although any one of the constraints could be deleted, we will n do so to preserve the incidence structure of the constraint matrix. Thus rewriting the constraints a $1x = b$, we have that the column of A corresponding to arc ij, A^{ij}, has a 1 in row i and a -1 in r j, while all other entries in the column are zero. Furthermore, we have $b_1 = F$, $b_m = -F$, $b_i = 0$, i = 2, . . ., $m-1$. Now the Lagrangian for (P) is

$$L(x, u) = 1/2 x^T Q x + u^T [Ax - b].$$

W wish to find the saddle point of $L(x, u)$, i.e., we wish to

$$\underset{u}{\text{Max}} \; \underset{x}{\text{Min}} \; L(x, u).$$

7 for any u, we can find the minimal value of x by differentiating $L(x, u)$ with respect to x:

$$x^T Q + u^T A = 0$$

$$u^T A = -x^T Q.$$

Substituting (6) into (5), we have

$$L(x, u)=1/2x^TQx+[-x^TQ]x-u^Tb,$$

or

$$L(x, u)=-1/2x^TQx-u^Tb,$$

or

$$L(x, u)=-\sum_{ij\epsilon I} q_{ij}(x_{ij})^2-\sum_{i=1}^{m} u_ib_i. \tag{7}$$

This form of the Lagrangian is closely related to Dorn's dual [3].

Now since A is an incidence matrix, the component of u^TA corresponding to the variable x_{ij} is (u_i-u_j). Clearly, the component of x^TQ corresponding to x_{ij} is $x_{ij}[2q_{ij}]$. Thus, by (6),

$$x_{ij}=[1/2q_{ij}]\,[u_j-u_i]. \tag{8}$$

Substituting (8) into (7) we have

$$-\sum_{ij\epsilon I} q_{ij}\{[1/2q_{ij}]\,[u_j-u_i]\}^2-\sum_{i=1}^{m} u_ib_i. \tag{9}$$

Now we wish to maximize (9) over u, that is,

$$\operatorname*{Max}_{u} -\sum_{ij\epsilon I} [1/4q_{ij}][u_j-u_i]^2-\sum_{i=1}^{m} u_ib_i. \tag{10}$$

Problem (10) is a concave, unconstrained, quadratic programming problem with m variables.

After solving problem (10), the optimal values u_i^* are used to compute the optimal flow x_{ij}^*, using (8). Thus problem (P), which had m constraints and n variables, can be solved by solvi an unconstrained problem with only m variables. Since a network always has more links than nod the unconstrained problem (10) has fewer variables than (P). This is especially important for lar network problems where the number of arcs is usually much greater than the number of nodes.

Since problem (10) is an unconstrained, concave, quadratic maximization problem, it can solved by setting the gradient equal to zero and solving the resulting system of linear equatio However, it is interesting to note the efficiency of the conjugate gradient algorithm for problem (1 Although it is not necessary to use the conjugate gradient technique, its computational succe warrants mentioning.

3. COMPUTATION RESULTS

Four sets of 10 minimum-cost flow problems were solved using FORTRAN IV on the CD Cyber 70, model 72. Each of the first 10 problems consisted of 25 nodes, the second set had nodes, the third had 75 nodes, and the fourth had 100 nodes. In each problem there was a li between each pair of nodes. The cost parameters, q_{ij}, for each link in the network were chos randomly as follows. Node 1 (the source) was assumed located at coordinates (0, 0), and node (the last node and destination) was given random coordinates (x_m, y_m), each in the interval (20, 120 The coordinates (x_i, y_i) of nodes i, $i=2, 3, \ldots, m-1$, were then generated randomly with $x_i\epsilon[0, x_m]$ and $y_i\epsilon[0, y_m]$. Next, q_{ij} was set equal to the distance between nodes i and j. Finally F, t flow required to be sent from node 1 to node m, was generated randomly in the interval (0, 20

Now the primal problem in each case has m (the number of nodes) conservation of flow constraints (one is redundant) and $m(m-1)/2$ variables. Thus each dual problem has m variables. It is well-known [6] that the conjugate gradient algorithm converges in m iterations for an unconstrained quadratic programming problem with m variables. Most nonlinear programming algorithms converge only in the limit, however, and use the termination criterion of stopping when functional values at two successive iterations differ by less than some epsilon. This stopping rule was used with $\epsilon = .001$ for each of the 40 problems solved. In almost every problem, the optimal value was in thousands or tens of thousands; thus the solution was accurate to less than 10^{-4} percent.

To test for roundoff error, after solving for the ϵ optimal dual solution, the dual variables were used to compute the optimal primal variables, x_{ij}^*, using (8). Primal feasibility in (2), (3), (4) was then checked, and numerical error was found to be almost nonexistent. Equality of the primal constraints held with less than .003 error in almost every case; the worst error noted was a discrepancy of .009 in equation (3). Next the primal objective function was computed from the primal variables, and was found to differ from the computed dual optimal value by less than .1 in every case. In all but a few cases, equality held to four decimal places. When the conjugate gradient algorithm was allowed to run m iterations, primal feasibility and primal-dual equality was exact in every case.

Average computing times (excluding the time to generate the random data) and numbers of iterations are given in Table 1. Note the remarkable fact that the number of iterations for an ϵ optimal solution to (P) was essentially unchanged from a 24 constraint, 300 variable primal problem to a 100 constraint, 4,950 variable problem.

TABLE 1

Problems	Number of nodes (constraints)	Number of arcs (variables)	Average number of iterations	Average CPU time (seconds)
1–10	25 (24)	300	10.7	0.3
11–20	50 (49)	1,225	10.6	1.0
21–30	75 (74)	2,775	11.0	2.3
31–40	100 (99)	4,950	11.2	4.0

CONCLUSION

We have examined a quadratic network problem and have shown that its structure can be exploited to yield an extremely efficient solution technique. It is hoped that the technique described will be useful not only for problems such as (P), but also as an approximation to problems more general than (P). For example, when using branch and bound for integer nonlinear programming problems, problem (P) may be solved to compute a lower bound. Also, problem (P) may be solved when using decomposition techniques to isolate special structure in more general problems.

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NEQUALITIES FOR MIXED INTEGER PROGRAMS WITH STRUCTURE

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ABSTRACT

Mixed integer programming problems arising in practice often contain special structures such as imbedded networks and multiple choice constraints. Easily derived inequalities are given that can be used to reduce the range of admissible solutions for such problems.

NTRODUCTION

Mixed integer programming problems that arise in practical applications frequently contain p ial constraint structures (e.g., imposed by bounded variables, network relations, or multiple h ce inequalities) which are easily identifiable yet which significantly influence the set of feasible l tions. Consequently, a solution strategy that utilizes derived inequalities (either as part of a u ing approach or a branch and bound approach) is enhanced by the ability to take advantage f ese special constraint structures to determine more restrictive conditions on the solution space.

In papers [3, 5, 6] the authors addressed the problem of generating convexity cut inequalities exploiting multiple choice structures. Here we specify new and broader classes of convexity that subsume the previous cuts of [3, 5, 6] ,and in addition make it possible to take advantage n arbitrary constraint structure without specifying that structure *a priori*. We also extend nteger strengthening procedures of [5] to apply to the new inequalities.

OTATION AND BASIC RESULTS

We provide the fundamental characterization of the new inequalities in this section, and then specific illustrative examples in section 3. To keep the notation and discussion as simple as ible, we will not reiterate the rationale underlying convexity cut inequalities or the details

of their generation (which are covered, for example, in [1, 2, 7], but will instead focus attention on identifying the special convex set R relative to which convexity cuts are defined. The form of this set automatically gives the updated form of the convexity cut for the current linear programming tableau representation, and thus its specification uniquely determines the resulting cuts.

Let s_j, $j \in J$, denote a collection of variables that are integer valued for every feasible solution to the mixed integer programming (MIP) problem. For example, if the standard integer variables are denoted x_i, $i \in I$, then each s_j may be given as some integer linear combination of these x_i plus an integer constant. The s_j variables are not required to be nonnegative. In the simplest applications s_j is one of the standard variables multiplied by ± 1. (Other useful identities for s_j are provided in the examples and subsequent discussion.) The value of s_j that results from the current linear programming (extreme point) solution to the MIP problem will be denoted s^*_j. We assume that this solution is not integer feasible (i.e., does not satisfy the integer restrictions for the MIP problem).

Then we may state our first result as follows:

THEOREM 1: Let u_j, $j \in J$, be integer constants and let a_j, $j \in J$, be positive coefficients for which the current linear programming solution to the MIP problem satisfies

$$s^*_j < u_j + 1, \quad j \in J \tag{1}$$

$$\sum_{j \in J} a_j s^*_j > \sum_{j \in J} a_j u_j. \tag{2}$$

Then a class of valid convexity cuts for the MIP problem which excludes the current linear programming solution is given by stipulating the convex set R to be the set of s_j, satisfying

$$s_j \leq u_j + 1, \quad j \in J \tag{3}$$

$$\sum_{j \in J} a_j s_j \geq \sum_{j \in J} a_j u_j. \tag{4}$$

PROOF: The justification is a direct consequence of the convexity cut lemma [2] by showing that the interior of R contains the linear programming solution but no feasible MIP solution. Conditions (1) and (2) explicitly accommodate the former. The latter is established by noting that a feasible MIP solution in the interior of R must satisfy $s_j \leq u_j$ for all $j \in J$, and hence (multiplying by the positive coefficients a_j and summing) must satisfy $\Sigma a_j s_j \leq \Sigma a_j u_j$. But all solutions in the interior of R must satisfy (4) as a strict inequality. Thus the theorem is proved by contradiction.

Whenever at least one s^*_j is not an integer in the linear programming solution, it is easy to determine values for the integers u_j that satisfy conditions (1) and (2), and thus give rise to an acceptable R (defined by (3) and (4)), regardless of the choice of the positive coefficients a_j, as noted in the following remark.

REMARK 1: Condition (1) holds if, and only if, the u_j are selected to satisfy $u_j \geq [s^*_j]$ (where the square brackets denote the greatest integer not exceeding the quantity inside). Moreover whenever some s^*_j is not an integer, condition (2) will automatically hold by taking $u_j = [s^*_j]$ for all $j \in J$ (although other values of u_j may also work).

A principal use of Theorem 1 occurs by replacing (4) with a second inequality which implies (4) relative to the constraints of the MIP problem. We will first state the result concerning the second inequality, and then give examples that clarify its use in the next section.

REMARK 2: Assume that all feasible solutions of the MIP problem satisfy the inequality

$$\sum_{j\in J} a_j s_j \geq w,$$

e w is an arbitrary variable, either integer or continuous (e.g., a linear combination of the ard integer and continuous variables of the MIP problem, plus a constant). Then (2) and (4) heorem 1 may respectively be replaced by

$$w^* > \sum_{j\in J} a_j u_j$$

$$w \geq \sum_{j\in J} a_j u_j.$$

t should be noted that every constraint equation of the MIP problem gives rise to two in- ities of the form (5). That is, if the constraint is

$$\sum_{i\in M} b_i x_i = b_0,$$

e writing this equation in its two inequality forms yields (5) with $w = b_0 - \sum b_i x_i + \sum a_j s_j$ and $w = -b_0 + \sum b_i x_i + \sum a_j s_j$. Assuming each s_j variable is a linear combination of the x_i variables a constant, then w is also, since, in particular, (6) implies $w = \sum a_j s_j$.

The class of cuts identified by Theorem 1 provides the basis for taking advantage of structure nly by Remark 2 (making use of a constraint such as (6)), but also in a stronger way. This is plished by identifying a larger convex set to take the place of the convex set R defined in rem 1, where this new set arises by disregarding some of the inequalities collectively given in id (4) (or (3) and (4′)). The rather simple theorem following is the general result; its corollary ually the form in which it is applied.

THEOREM 2: If any subset of the inequalities in (3) and (4) (or (3) and (4′)) serves to ate an R whose interior contains no feasible MIP solutions, then this (larger) R may be used id in Theorem 1.

PROOF: In the proof of Theorem 1, it was shown that the original (smaller) R contained the programming solution in its interior. Therefore, the same may be said of the larger R. The condition needed for the convexity cut lemma, that the interior contains no feasible MIP on, is given as an assumption of Theorem 2.

COROLLARY 1: If any of the inequalities in (3) and (4) (or on (3) and (4′)) are satisfied as inequalities by all feasible MIP solutions, then such inequalities may be dropped in the tion of R in Theorem 1.

PROOF: The full set of inequalities generate an R whose interior contains no feasible *MIP* ons (as shown in the proof of Theorem 1). Enlarging this R by dropping some of the defining raints that are satisfied strictly by *all MIP* solutions cannot "add" any *MIP* solutions to the or. Corollary 1 then follows by Theorem 2.

Before giving specific examples of uses of these results, we will indicate how the previous cuts [4, 5] can be obtained as special cases of those characterized by Theorems 1 and 2.

REMARK 3: The multiple choice cuts of [5] are obtained from Theorem 1, using (4 i
place of (4), for the case in which each coefficient $a_j=1$, and Equation (6) (from which the two f·m
of (4′) are obtained) is the multiple choice constraint $\sum_{i\epsilon M} x_i=k$.

REMARK 4: The multiple choice "hyperplane" cuts of [3] for the 0–1 *MIP* problem r·l
from Theorem 1 in the manner specified in Remark 3, except that R is enlarged by dropping
as justified in this case by Corollary 1.

In the applications summarized by the two preceding remarks, condition (2) specializes t

$$\sum_{j\epsilon J} (u_j-[s^*_j])<\sum_{j\epsilon J} f^*_j$$

where the f_j^* are the "fractional parts" of the linear programming solution. Thus $\sum f_j^*$ provi
measure of the "slack" available for assigning values to the u_j.

REMARK 5: The Gomory mixed integer cuts of [4] result from Theorem 1 for the sp·i
case in which there is only one integer variable s_j, where $u_j=[s^*_j]$, and where (3) and (4) reduc t
$[s^*_j]\leq s_j\leq[s^*_j]+1$. In order to get the smallest possible coefficients in the resulting cut, s_j is def·e
as a linear combination of one basic integer variable which is currently fractional and sev·
nonbasic integer variables. See section 4 for details of this strengthening procedure.

3. NEW USES OF THE THEOREMS, WITH ILLUSTRATIONS

We will now show how to exploit the foregoing results for the situation in which Equation 6
is an arbitrary constraint of the MIP problem.

A natural (and particularly simple) way to take advantage of (6) is the following. Let b
any subset of M for which x_j is integer valued (and $b_j\neq 0$) for all $j\epsilon J$. Then letting $s_j=x_j$ if b_j
and letting $s_j=-x_j$ if $b_j<0$, we obtain

$$\sum_{j\epsilon J} b_j s_j\geq b_0-\sum_{M-J} b_i x_i \tag{7}$$

as one of the two inequalities implied by (6), whereupon the positive coefficients a_j can simply
selected to be $a_j=|b_j|$, $j\epsilon J$, and w can be given by $w=b_0-\sum_{M-J} b_i x_i$. Thus (7) has exactly the form
(5). For the same choice of J, but selecting the variables s_j to be the negative of those in (7),
obtain

$$\sum_{j\epsilon J} |b_j|s_j\geq -b_0+\sum_{M-J} b_i x_i, \tag{8}$$

which gives the counterpart inequality to take the role of (5), where the a_j coefficients are the sa
as before and where $w=-b_0+\sum_{M-J} b_i x_i$. In spite of the fact that the a_j coefficients are the sa
the cuts to be generated by reference to (7) and (8) are characteristically different.

EXAMPLE 1: Consider the equation

$$3x_1+4x_2-8\tfrac{1}{2}x_3-5x_4+7\tfrac{1}{2}x_5-2x_6+1\tfrac{1}{5}x_7=2\tfrac{8}{15},$$

vere the values of the variables in the linear programming solution are given by the vector (1/3 2; 1½ 2/3 1/2 2⅕ 13). Suppose that $x_1, \ldots, x_5$ are integer variables and x_6, x_7 are continuous. Letting J be the index set for the integer variables, the inequality (7) becomes

$$3s_1+4s_2+8\tfrac{1}{2}s_3+5s_4+7\tfrac{1}{2}s_5 \geq 2\tfrac{3}{15}+2x_6-1\tfrac{1}{5}x_7,$$

vere $s_j=x_j$ for $j=1, 2, 5$, and $s_j=-x_j$ for $j=3, 4$. Defining w to be the right hand side of this inequality, the inequality assumes the form of (5), with $w^*=-8\frac{3}{3}$. Thus, by (2′), the integers u_j that we seek to define a legitimate cut must satisfy

$$-8\tfrac{3}{3} > 3u_1+4u_2+8\tfrac{1}{2}u_3+5u_4+7\tfrac{1}{2}u_5.$$

In addition, by (1) (and Remark 1), these integers must also satisfy $u_1 \geq 0$, $u_2 \geq 0$, $u_3 \geq -2$, $u_4 \geq -1$, $u \geq 0$. Thus, some possible values for the u_j are given by the vectors $(u_1, u_2, u_3, u_4, u_5)=(0\ 0\ -2\ 1\ 0)$, $(0\ 1\ -1\ -1\ 0)$, $(1\ 0\ -2\ 0\ 0)$, and $(1\ 0\ -2\ -1\ 1)$. Corresponding to each of these vectors, w is required by (4′) to be bounded from below respectively by -22, $-9\frac{1}{2}$, -14, $-11\frac{1}{2}$. The upper bounds on the variables s_j, from (3), are given by increasing each of the components of the foregoing vectors by 1.

EXAMPLE 2: To illustrate the strengthening that is possible by Theorem 2 for bounded variable problems, suppose in the preceding example that x_1, x_2 and x_5 are 0–1 variables, and x_3 and x_4 are constrained to be nonnegative. Then the upper bounds on the s_j variables (obtained by adding 1 to the components of the u_j vectors) yield strictly redundant inequalities in the last three vectors, and these may be dropped to provide larger convex regions to define the cuts. Specifically, the inequalities that may be dropped are: $s_2 \equiv x_2 \leq 2$ from the second vector; $s_1 \equiv x_1 \leq 2$ and $s_4 \equiv -x_4 \leq 1$ (i.e., $x_4 \geq -1$) from the third vector; $s_1 \equiv x_1 \leq 2$ and $s_5 \equiv x_5 \leq 2$ from the fourth vector. It should be noted that although these inequalities are strictly redundant from the standpoint of the feasible solution set, they are *not* redundant relative to the region defining the convexity cut, and thus their removal indeed gives rise to deeper cut inequalities.

When the MIP problem contains more "elaborate" structure than bounds on variables, it is possible to exploit (6) still more fully by means of Theorem 2. One way to accomplish this is by aggregating variables. For example, suppose that in equation (6) there is a subset of the integer variables x_i, say $i \in Q$, for which it is possible to obtain a restrictive bound on $\sum_{i \in Q} b_i x_i$, where the coefficients b_i are integers. Then, instead of defining $s_j=\pm x_j$ for each of the integer variables, it may be more useful to define one of the s_j to equal $\sum_{i \in Q} b_i x_i$, thereby enlarging the convex region, as illustrated by the following example.

EXAMPLE 3: Suppose that in Example 2 the variables x_1 and x_2 are required to satisfy the multiple choice constraint $x_1+x_2 \leq 1$. This implies that $3x_1+4x_2$ is bounded above by 4. Thus, defining $s_1=3x_1+4x_2$, and defining the remaining s_j, $j \geq 3$, as before, we obtain the inequality

$$s_1+8\tfrac{1}{2}s_3+5s_4+7\tfrac{1}{2}s_5 \geq 2\tfrac{3}{15}+2x_6-1\tfrac{1}{5}x_7.$$

Thus the integers u_j must satisfy

$$-8\tfrac{3}{3} > u_1+8\tfrac{1}{2}u_3+5u_4+7\tfrac{1}{2}u_5,$$

u_3, u_4 and u_5 are bounded from below as before, and $u_1 \geq 3s_1^* + 4s_2^* = 1 + 2\frac{2}{3} = 3$. Note that one set values for the new u_j that satisfy these conditions is given by the vector (4 −2 −1 1). The resulti upper bounds on the s_j (from (3)) are strictly redundant for all $s_3 \leq -1$ and $s_4 \leq 0$. This yields a c that was not available in Example 2 where the constraint $x_1 + x_2 \leq 1$ was not exploited, since it w not possible in that example to reduce the set of bounding inequalities on the s_j to these two.

When dealing with the pure integer programming problem, or with any constraint of the MI problem in which no continuous variables appear, it is possible to obtain deeper cuts because the fact that w can always be made to be a constant, whereupon (4′) is strictly redundant b virtue of (2′) where $w = w^*$. Consequently, the only inequalities defining the convex region a those of (3).

EXAMPLE 4: In Examples 1, 2, and 3, suppose that x_6 and x_7 are no longer present (so tha all variables are integer valued) and the constant term of the equation is $-8\frac{2}{3}$ (as occurs by treatin x_6 and x_7 as constants with the same values as before). Then $w = w^* = -8\frac{2}{3}$, yielding the restrictio on the u_j and the s_j, $j \leq 5$, as in the preceding examples, but where the convex region is enlarged b the absence of a constraint on w.

There are a variety of other ways to apply Theorems 1 and 2 in the setting of the MIP proble The foregoing is intended merely to provide an illustrative sampling.

4. ADDITIONAL STRENGTHENING PROCEDURES

For cuts defined by reference to an inequality such as (5), it is possible to determine modifie definitions of the s_j variables that will allow the edges of the linear programming cone to be extende deeper into the region R and thus yield stronger cuts. However, these modified definitions ma result in sacrificing the ability to identify strictly redundant bounds on some of the s_j, and cons quently the strengthening of the modified definitions may be at the expense of strengthening b means of Theorem 2.

The modified definitions are achieved as follows. Consider the s_j in "updated equation form in the current LP tableau. Then, as a particular edge of the cone is extended, the values of the variables are given by equations of the form $s_j = s^*_j + \alpha_j\theta$, where θ is the nonbasic variable corre sponding to the given edge. If θ is an integer variable, and a particular variable s_k blocks the edg extension by (3) (i.e., θ is prevented from increasing beyond the value θ^*, because $u_k + 1 = s^*_k + \alpha_k\theta^*$) then the variable s_k can be made "nonblocking" by replacing it with the variable $\bar{s}_k = s^*_k + (\alpha_k - r_k)\theta$ and the bound on $\bar{s}_k$ is not achieved at θ^*; i.e., $u_k + 1 > s^*_k + (\alpha_k - r_k)\theta^*$. For this modified definitio of s_k to be legitimate, one or more of the other s_j variables must also be similarly redefined, replacing them by variables $\bar{s}_j = s_j - r_j\theta$, where, for these variables, the integers r_j are negative. For convenience we will in fact suppose all of the s_j are thus redefined, allowing $r_j = 0$ for those whose identity w do not elect to change. Then there are two restrictions to observe for the redefined variables. The first is

$$\sum_{j \in J} \alpha_j r_j \leq 0,$$

since as long as this holds then (5) will continue to be satisfied with s_j replaced by $\bar{s}_j$. The second restriction is that the selection of the integers r_j must not create a new variable that is blocking a (or before) $\theta = \theta^*$. That is, $u_j + 1 > s^*_j + (\alpha_j - r_j)\theta^*$ must hold for all j. Having determined acceptable

lues for the r_j at θ^*, the process may be repeated as subsequent variables (s_j) block the edge ex- sion (modifying previous values assigned to the r_j) until it is impossible or requires undue effort modify any of the r_j without violating one of the indicated restrictions. The same type of deepen- of the edge extensions can be carried out independently for each edge that corresponds to an eger variable.

Whether this type of strengthening will be superior to the type of strengthening available from eorem 2 will depend on the problem structure and the on number of nonbasic variables that are eger valued. It should be noted, however, that these two types of strengthening need not be mu- lly exclusive. In particular, it is completely legitimate to apply the procedure of this section only those s_j that would not be dropped by Theorem 2 (and ignoring these latter s_j as possible block- variables), thereby producing a cut that is always at least as deep as, and generally deeper n, that resulting from Theorem 2 alone.

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ON A CONSTRAINT QUALIFICATION IN A NONDIFFERENTIABLE PROGRAMMING PROBLEM*

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ABSTRACT

A mathematical programming problem with an objective function containing the square root of a positive semidefinite quadratic form has been considered by Mond. In order to use a transposition theorem of Eisenberg, Mond introduces a complicated constraint qualification. In this note we give a simple geometric characterization to this constraint qualification and show that it is implied by the generalized Slater constraint qualification.

INTRODUCTION

The following problem has been discussed by Mond [4]:

$$\text{Minimize } F(x)=f(x)+(x^tBx)^{1/2}$$

$$\text{subject to } g(x)\geq 0$$

where f and g are differentiable functions from R^n into R and R^m, respectively, and B is a positive semidefinite $n\times n$ symmetric matrix. In spite of the nondifferentiability of the objective function F, Mond derives a necessary condition for optimality very much like Kuhn-Tucker conditions under an hypothesis which is in the nature of a constraint qualification. This hypothesis is made so that a transposition theorem of Eisenberg [1, 2], may be used. In this note we examine this hypothesis and show how it may be interpreted geometrically. From this interpretation we show that the generalized Slater constraint qualification implies this hypothesis.

*The research reported in this paper was performed while the authors were visiting professors at the Department of Applied Mathematics, Technion-Israel Institute of Technology, Haifa, Israel.

DIRECTIONAL DERIVATIVES

For a function h from R^n into R, a point x_0 in R^n, and vector z, the directional derivative of h in the direction z at the point x_0, which we denote by $h'(x_0;z)$, is defined by

$$h'(x_0; z)=\lim_{\tau\to 0^+} \tau^{-1}[h(x_0+\tau z)-h(x_0)] \tag{1}$$

where $\tau\to 0^+$ means τ approaches zero through positive values. If h is differentiable at x_0 and its gradient there is denoted by $\nabla h(x_0)$ then

$$h'(x_0; z)=z^t\nabla h(x_0). \tag{2}$$

LEMMA: Let F be the objective function of our problem. Then

$$F'(x_0; z)=z^t\nabla f(x_0)+z^tBx_0/(x_0^tBx_0)^{1/2} \quad \text{if } x_0^tBx_0>0,$$

$$F'(x_0; z)=z^t\nabla f(x_0)+(z^tBz)^{1/2} \quad \text{if } x_0^tBx_0=0.$$

PROOF: F is differentiable at each point x_0 where $x_0^tBx_0\neq 0$ and at such a point

$$\nabla F(x_0)=\nabla f(x_0)+Bx_0/(x_0^tBx_0)^{1/2},$$

and this formula together with (2) already establishes the first part of the lemma. Now suppose $x_0^tBx_0=0$. Then $Bx_0=0$. This may be seen from the generalized Cauchy-Schwartz inequality which gives for any x

$$(x^tBx_0)^2\leq(x^tBx)(x_0^tBx_0)=0,$$

and since this holds for any x we deduce $Bx_0=0$. Therefore,

$$F(x_0+\tau z)-F(x_0)=f(x_0+\tau z)-f(x_0)+[(x_0+\tau z)^tB(x_0+\tau z)]^{1/2}$$

$$=f(x_0+\tau z)-f(x_0)+\tau(z^tBz)^{1/2}.$$

Dividing by τ and taking the limit as $\tau\to 0^+$ gives the second assertion of the lemma.

Following [4], let x_0 be a feasible point and let Q_0 be the active constraint index set, i.e.,

$$Q_0=\{i|g_i(x_0)=0\}.$$

Then we define

$$Z_0=\{z|z^t\nabla g_i(x_0)\geq 0\forall i\in Q_0 \text{ and } F'(x_0; z)<0\}.$$

Of course, in [4] this definition is given in terms of the formula which appears in the preceding lemma.

THE SLATER CONDITION

In [4] a necessary condition for the feasible point x_0 to be optimal is derived undert he condition that Z_0 is empty. We show that this condition is satisfied by any feasible optimal point if there exists a feasible point which satisfies the nonaffine constraints as a strict inequality. This is just the generalized Slater constraint qualification. A number of other classical constraint qualifications among those discussed, for example in [3], also yield the result that Z_0 is empty if x_0 is optimal.

THEOREM: Suppose that g_i is concave for each i, and that there exists a feasible point x_* that $g_i(x_*)>0$ whenever g_i is nonaffine. Then if x_0 is an optimal feasible point, Z_0 is empty.

PROOF: First note from (1), the definition of directional derivative, that for any function h

$$h'(x_0; z)>0 \text{ implies } h(x_0+\tau z)\geq h(x_0) \text{ for } \tau$$

negative and sufficiently small. If h is differentiable, the same conclusion follows from $z^t\nabla h(x_0)>0$ use of (2). If h is affine, then $z^t\nabla h(x_0)\geq 0$ implies $h(x_0+\tau z)\geq h(x_0)$ for all $\tau>0$ because in this $h(x)=x^t\nabla h(x_0)$ plus a constant.

Now suppose that x_0 is an optimal feasible point and that z lies in Z_0. Let $z_*=x_*-x_0$. Then we show that for some positive ρ the following are satisfied:

(i) $(z+\rho z_*)^t\nabla g_i(x_0)>0$ $\forall i \in Q_0$ such that g_i is nonaffine,

(ii) $(z+\rho z_*)^t\nabla g_i(x_0)\geq 0$ $\forall i \in Q_0$ with g_i affine,

(iii) $x_0+\tau(z+\rho z_*)$ is feasible for ρ positive and sufficiently small,

(iv) $F'(x_0; z+\rho z_*)<0$.

Once ρ is selected satisfying (i) and (ii), (iii) is automatically satisfied because by the preceding ment (i) implies that $x_0+\tau(z+\rho z_*)$ satisfies for i in Q_0 the nonaffine constraints for τ sufficiently l, and (ii) implies that this point satisfies the affine constraints for i in Q_0 for any positive τ. o i not in Q_0, $g_i(x_0)>0$, and since g_i is continuous, $g_i(x_0+\tau(z+\rho z_*))\geq 0$ for τ sufficiently small.

Now we will show that these four conditions may be satisfied. From the definition of Z_o, $_0; z)<0$. From the lemma we see that F' is continuous in Z, hence (iv) is satisfied if ρ is sufficiently l. Now suppose i is in Q_0 and g_i is affine. Say $g_i(x)=x^tc+d$ for some vector c and number d. h

$$\nabla g_i(x_0)=c \text{ and } x_0^tc+d=0.$$

$$z_*^t\nabla g_i(x_0)=x_*^tc-x_0^tc=x_*^tc+d=g_i(x_*)\geq 0,$$

since $z^t\nabla g_i(x^0)\geq 0$ by definition of Z_0, (ii) is established. To show (i) we use the fact that g_i is ave. Thus

$$z_*^t\nabla g_i(x_0)=(x_*-x_0)^t\Delta g_i(x_0)\geq g_i(x_*)-g_i(x_0)=g_i(x_*)>0,$$

again, $z^t\nabla g_i(x_0)\geq 0$, thus establishing (i).

Finally, we get a contradiction from (iii) and (iv), because these imply, using the definition of tional derivative that for τ positive and sufficiently small:

$$x_0+\tau(z+\rho z_*) \text{ is feasible and}$$

$$F(x_0+\tau(z+\rho z_*))<F(x_0)$$

since x_0 is an optimal point, this is a contradiction.

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ZERO-SUM GAMES WITH MULTIPLE GOALS*

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ABSTRACT

This paper investigates a class of two-person zero-sum multiple payoff games in which each component is deterministic. The problem of minimizing a player's total underachievement of a fixed set of goals is shown to reduce to a linear programming problem. Applications are discussed and illustrative examples are presented.

INTRODUCTION

In this paper we consider a class of games in which multiple goals or objectives are present. Numerous examples of such problems exist, particularly in bidding and advertising industries. As an example, consider a firm or government agency which is about to submit bids on a set of contracts, some of short and some of long-run duration. The winning of a specific contract or subset of the contracts can involve many aspects other than the actual dollar value being submitted by the firm and its competitors. Such single criterion models have been considered, for example, in [3] and [7]. Another important consideration, not allowed for in these models, is that of obtaining a portfolio of contracts which is diversified enough to allow for a minimum of unemployment in different trades. Other considerations are maintaining an equitable market share in different areas with different customers, and maintaining an even balance between inflow and outflow of resources over the period of contract tenure.

In competitive advertising, each firm generally has a fixed budget to work against. In the case of single firm allocating advertising funds in a market in which only a single major competitor exists, it is reasonable to model the problem as a zero-sum game. Primarily, the objective of a particular firm is to gain as large a market share as possible from its competitor. Also of concern to the firm, however, is that a variety of media be selected with the aim of reaching a diversified segment of the population. These two goals may conflict in the short run but not necessarily in the long run. Therefore, problems such as these which fit into the realm of zero-sum game theory involve multiple rewards or goals.

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The concept of multiple payoff games was introduced by Blackwell [3], and in more depth la by Contini [5]. Contini examines two models of random games with vector payoffs namely the pected value model and the probability maximization model. Since the opponents strategy is sumed known, both models reduce to the usual vector valued linear optimization problem. Zele [11] examines vector valued two-person zero-sum games and shows that by introducing a parame vector $\lambda=(\lambda_1, \ldots, \lambda_m)$, the game reduces to a parametric linear programming problem. He th discusses the concept of Pareto Optimal solutions and ideal points for vector valued games. Aub [1] describes a class of "hierarchical games" in which the objective of the decisionmaker or cent unit of an organization is to choose a strategy which minimizes the total squared deviation of t payoffs to subunits from the values of the games for these subunits. Aubin suggests the gradie algorithm as a solution method for such minimax problems.

In section 2 of this paper we consider two-person zero-sum deterministic games with multi goals. We assume that a goal relative to each attribute has been specified by firm 1 (player 1) a that player 1 adopts the criterion of minimizing the total expected underachievement of these goa The resulting problem is shown to reduce to a linear programming problem. Examples illustrati the results are included.

2. DETERMINISTIC MULTIGOAL ZERO-SUM GAMES

Consider a two-person zero-sum game in which for each pure strategy $i \in \{1, \ldots, m\}$ f player 1 and $j \in \{1, \ldots, n\}$ for player 2 there is a vector of payoffs $\alpha_{ij}=(\alpha_{ij}^1, \ldots, \alpha_{ij}^K)$ fro player 2 to player 1 from K attributes. Let $\{g^k\}_{k=1}^K$ be a set of goals or satisfaction levels (o for each attribute) specified by player 1, and assume the player wishes to choose a strategy f which his long-run average payoff from each attribute k is at least g^k. (Actually, the g^k may chosen either by player 1 or jointly determined by the two players. Generally, a firm cannot assi values to its competitors' goals and must, therefore, concentrate on its own.)

Let us assume player 1 selects his pure strategy i with probability x_i and player 2 selects with probability y_j. Denote the strategy spaces for the players by

$$\mathcal{X}=\{X=(x_1, \ldots, x_m) | \sum_i x_i=1, x_i \geq 0 \forall i\}^*$$

and

$$\mathcal{Y}=\{Y=(y_1, \ldots, y_n) | \sum_j y_j=1, y_j \geq 0 \forall j\},$$

respectively. The objective of player 1 is then to select a strategy X^0 such that if player 2 selec Y^0 the expected payoff from the *kth* attribute is at least g^k. Letting A^k denote the matrix whose element is α_{ij}^k, this implies that

$$X^0 A^k Y^0 \geq g^k.\dagger$$

---

*The notation $\sum_i$ shall denote summation over the entire index set of i. For example $\sum_i x_i = \sum_{i=1}^m x_i$.

†We consider in this paper the one-sided goal problem since in most applied problems goals can be defined values one wishes to be "at or above." Moreover, any two-sided goal can be expressed as a pair of one-sided goals.

Generally, no mixed strategy X^0 exists for player 1 which will "guarantee" that all goals can ttained. In this situation many different approaches can be taken in defining an "optimal egy." One approach is to rank the goals in order of preference and then select a strategy which guarantee (under the maximin criterion) that as many of the goals as possible, in the specified , are satisfied. An alternate approach is to adopt that strategy which maximizes the number als attained regardless of any *a priori* ordering. In both cases undesirable situations are likely ise. Player 1 would generally end up choosing a strategy which would guarantee that a small ber of goals were met at the expense of the others suffering gross underachievement. That is, er 1 could not guarantee that his competitor would not concentrate on preventing him from ving the very goals at which he is aiming.

A criterion which seems to be the most desirable (from player 1's standpoint) of those available, choose that strategy for which the total underachievement of goals is as small as possible, rdless of the opponent's strategy. To take account of the possible relative importance of , appropriate weights can be assigned. In this way the attempted attainment of a particular does not jeopardize the value of others. Moreover, if the game is zero-sum in each payoff onent, and if player 1 wishes to minimize the total underachievement of his goals, the e game is zero-sum to the extent that the opponent will attempt to maximize player 1's achievement.

Letting $\{w^k\}_{k=1}^K$ denote the relative weights attached to the K goals, player 1's problem then mes that of determining $X^* \epsilon \mathcal{X}$, which solves

$$\min_{X\epsilon\mathcal{X}} \max_{Y\epsilon\mathcal{Y}} \min_{\{d^k\}} \sum_k w^k d^k,$$

ject to

$$XA^kY + d^k \geq g^k,\ d^k \geq 0 \ \forall\, k.$$

operator "$\min_{\{d^k\}}$" is required here to ensure that the d^k chosen actually measure the underachieve- of the g^k. In the absence of this the d^k would be interpreted mathematically as being under the rol of player 2, thereupon rendering (2.1) unbounded.

We prove the following:

THEOREM 2.1: Problem (2.1) is equivalent to a linear programming problem.

PROOF: Let us write the constraints of (2.1) in the form

$$d^k \geq X\hat{A}^kY$$

e

$$\hat{A}^k \equiv (\hat{a}_{ij}{}^k) = (g^k - a_{ij}{}^k).$$

For any given pair of strategies X^0 and Y^0, (2.1) is, then, a linear programming problem whose is given by

$$\max_{\alpha\epsilon\Psi} \sum_k (X^0\hat{A}^kY^0)\alpha^k,$$

e

$$\Psi = \{\alpha = (\alpha^1, \ldots, \alpha^k) \mid 0 \leq \alpha^k \leq w^k\}.$$

Since the duality is valid for all (X^0, Y^0), (2.1) is, therefore, equivalent to

$$\min_{X\epsilon\mathcal{X}} \max_{Y\epsilon\mathcal{Y}} \max_{\alpha\epsilon\Psi} \sum_k (X^k \hat{A} Y)\alpha^k. \tag{2.2}$$

To show the equivalence of (2.2) to a linear program, it is sufficient to write α in the fo

$$\alpha = \sum_{r=1}^{R} \gamma_r \alpha_r \tag{2.3}$$

with

$$\sum_{r=1}^{R} \gamma_r = 1, \ \gamma_r \geq 0 \ \forall r,$$

where

$$\{\alpha_r\}_{r=1}^{R}$$

denotes the set of extreme points of Ψ. If we make the transformation

$$z_{jr} \equiv y_j \gamma_r,$$

and note that

$$\sum_{jr} z_{jr} = 1,$$

(2.2) then reduces to the two-person game formulation

$$\min_{X\epsilon\mathcal{X}} \max_{Z} XAZ \tag{2.4}$$

where

$$A = (A(1), \ldots, A(r), \ldots, A(R)) \text{ and } A(r) = \sum_k \hat{A}^k \alpha_r{}^k$$

and where the maximization is taken over all Z in the $n \times R$ dimensional unit simplex. Sin (2.4) is in the standard two-person zero-sum game format, X^* can be determined by solving t equivalent linear programming problem

$$\min_{X\epsilon\mathcal{X}, \beta} \beta$$

subject to

$$\sum_i x_i A_{ij}(r) - \beta \leq 0 \forall j, r; X \epsilon \mathcal{X}, \tag{2.5}$$

where $A_{ij}(r)$ is the ijth component of $A(r)$. This proves the theorem.

It must be pointed out that a particular optimal solution Z^* to (2.4) may not be decomposab into probability distributions Y^* and γ^* for which $z^*{}_{jr} \equiv y_r{}^* \gamma_j{}^* \forall j, r$; although there will exist son optimal vector $\hat{Z}$ (not necessarily basic) for which such a decomposition can be brought abou The determination of $\hat{Z}$ is not important, however, since player 1 is interested only in finding X

Thus, the optimal mixed strategy X^* to (2.1) can be determined by solving the simpler line programming problem (2.5). Problem (2.5) can, moreover, be generalized to include a set of line restrictions of the form $BX \leq b$ on player 1's strategy space. Similar constraints $CY \leq c$ can imposed upon player 2's space as well, although the above transformation would have to be writt as

$$z_{lr} = \lambda_l \gamma_r$$

where

$$Y = \sum_{l=1}^{L} \lambda_l Y^l$$

$$\sum_{l=1}^{L} \lambda_l = 1, \qquad \gamma_l \geq 0 \forall l,$$

with $\{Y^l\}_{l=1}^{L}$ denoting the extreme point of the Y-space. $A_{ij}{}^r$ would need to be redefined accordingly.

Generating the convex hull of a general polyhedron can prove computationally difficult. Thus, the Y^l are not easily determined. The early work by Balinski [2], for example, is inefficient for any but small problems. More recent techniques developed by Manas and Nedoma [10], as well as those by Burdet [4], Gal [8] and Gal and Nedoma [9], have provided some new insights.

The extreme points of ψ can, however, be completely characterized.

THEOREM 2.2: The extreme points of ψ are given by

$$\{\alpha^r\}_{r=1}^{2^K}$$

where

$$\alpha^r = (\delta^1\omega^1, \delta^2\omega^2, \ldots, \delta^K\omega^K)$$

with

$$\delta^k \in \{0, 1\} \forall k \in \{1, 2, \ldots, K\}.$$

It is interesting to consider the two-person multigoal game as viewed by player 2. If player 2 adopts the strategy of attempting to maximize player 1's total underachievement of the goals $\{g^k\}_{k=1}^{K}$, as assumed at the beginning of this section, the problem player 2 should solve is, then,

$$\max_{Y \in \mathcal{Y}} \min_{X \in \mathcal{X}} \min_{\{d^k\}} \sum_k w^k d^k$$

subject to

$$XA^kY + d^k \geq g^k, \; d^k \geq 0 \; \forall k. \tag{2.6}$$

It is noted that (2.1) and (2.6) are not duals, however, in that (2.6) is equivalent to a quadratic rather than a linear programming problem.

We now present two examples to illustrate the results of this section.

EXAMPLE 1: Consider a multigoal game in which each player has two alternatives with three different payoffs going to those players. Specifically, let the goals be $g^1=1$, $g^2=5$, $g_3=4$ (measured in dollars, sales volume, or other representative units), and let the corresponding payoff matrices be given by

$$A^1 = \begin{bmatrix} 1 & -6 \\ 2 & -7 \end{bmatrix}, \quad A^2 = \begin{bmatrix} -10 & 7 \\ 3 & -4 \end{bmatrix}, \quad A^3 = \begin{bmatrix} 2 & 4 \\ -6 & -1 \end{bmatrix}$$

Let weights $\omega^1=2$, $\omega^2=0.5$, $\omega^3=1$ be assigned to goals g_1, g_2, and g_3 respectively.

The extreme points of Ψ are, then, given by

$$\alpha_1=(0, 0, 0), \quad \alpha_2=(2, 0, 0), \quad \alpha_3=(0, 0.5, 0), \quad \alpha_4=(0, 0, 1),$$

$$\alpha_5=(2, 0.5, 0), \quad \alpha_6=(2, 0, 1), \quad \alpha_7=(0, 0.5, 1), \quad \alpha_8=(2, 0.5, 1)$$

$$A(1) = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \quad A(2) = \begin{bmatrix} 0 & 14 \\ -2 & 16 \end{bmatrix} \quad A(3) = \begin{bmatrix} 7.5 & -1 \\ 1 & 4.5 \end{bmatrix}$$

$$A(4)=\begin{bmatrix}2 & 0\\ 10 & 5\end{bmatrix} \quad A(5)=\begin{bmatrix}7.5 & 13\\ -1 & 20.5\end{bmatrix} \quad A(6)=\begin{bmatrix}2 & 14\\ 8 & 21\end{bmatrix}$$

$$A(7)=\begin{bmatrix}9.5 & -1\\ 11 & 9.5\end{bmatrix} \quad A(8)=\begin{bmatrix}9.5 & 13\\ 9 & 25.5\end{bmatrix}$$

Solving the 15-constraint L.P. (2.5) on the MPS/360 computer package yielded the following results after 5 iterations:

$$x^*_1=1,\ x^*_2=0,\ \beta\simeq 14.$$

That is, there is a solution in pure strategies. It can be shown that $y^*_1=0$, $y^*_2=1$ is mathematically equivalent to an optimal strategy choice for the second player. In this case there is underachievement in the first goal only, meaning that

$$d^*_1=7,\ d^*_2=d^*_3=0.$$

EXAMPLE 2: In this example we choose three goals $g^1=4$, $g^2=1$, $g^3=2$ with $\omega^1=1$, $\omega^2=2$, $\omega^3=2.5$, and assume that each player has three alternatives. Let the payoff matrices be

$$A^1=\begin{bmatrix}2 & 5 & 1\\ -1 & -2 & 6\\ 0 & 3 & -1\end{bmatrix}, \quad A^2=\begin{bmatrix}-3 & 7 & 2\\ 0 & -2 & 0\\ 3 & -1 & -6\end{bmatrix}, \quad A^3=\begin{bmatrix}8 & -2 & 3\\ -5 & 6 & 0\\ -3 & 1 & 6\end{bmatrix}.$$

Constructing the α_r as done previously, we get the following matrices $A(r)$:

$$A(1)=\begin{bmatrix}0 & 0 & 0\\ 0 & 0 & 0\\ 0 & 0 & 0\end{bmatrix} \quad A(2)=\begin{bmatrix}2 & -1 & 3\\ 5 & 6 & -2\\ -4 & 1 & 5\end{bmatrix} \quad A(3)=\begin{bmatrix}8 & -12 & -2\\ 2 & 6 & 2\\ -4 & 4 & 14\end{bmatrix}$$

$$A(4)=\begin{bmatrix}-15 & 10 & -2.5\\ 17.5 & -10 & 5\\ 12.5 & 2.5 & 10\end{bmatrix} \quad A(5)=\begin{bmatrix}10 & -13 & 1\\ 7 & 12 & 0\\ -8 & 5 & 19\end{bmatrix}$$

$$A(6)=\begin{bmatrix}-13 & 9 & 0.5\\ 22.5 & -4 & 3\\ 8.5 & 3.5 & 15\end{bmatrix} \quad A(7)=\begin{bmatrix}-7 & -2 & -4.5\\ 19.5 & -4 & 7\\ 8.5 & 6.5 & 24\end{bmatrix}$$

$$A(8)=\begin{bmatrix}-5 & -3 & -1.5\\ 24.5 & 2 & 5\\ 4.5 & 7.5 & 29\end{bmatrix}$$

Solving the 22-constraint L.P. problem (2.5) using these matrices yields the following results:

$$x^*_1=.636024, \quad x^*_2=.157764, \quad x^*_3=.206211, \quad \beta^*=5.8149$$

A strategy sequence which is equivalent to an optimal choice for the second player is $y^*_2=1$, $y^*_1=y^*_3=0$. In this case goals one and three are not achieved.

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SOLVING QUADRATIC ASSIGNMENT PROBLEMS WITH RECTANGULAR DISTANCES AND INTEGER PROGRAMMING

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ABSTRACT

The problem considered involves the assignment of n facilities to n specified locations. Each facility has a given nonnegative flow from each of the other facilities. The objective is to minimize the sum of transportation costs.

Assume these n locations are given as points on a two-dimensional plane and transportation costs are proportional to weighted rectangular distances. Then the problem is formulated as a binary mixed integer program. The number of integer variables (all binary) involved equals the number of facilities squared. Without increasing the number of integer variables, the formulation is extended to include "site costs." Computational results of the formulation are presented.

1 THE PROBLEM

The quadratic assignment problem of Koopmans and Beckmann [9] can be described as follows: The problem is the assignment of n facilities to n specified locations. Each facility needs a given flow from each of the other facilities. The objective is to minimize the sum of transportation costs. Review articles of solution procedures on various forms of the quadratic assignment problem have been presented by Pierce and Crowston [11], and Hanan and Kurtzberg [4].

2 INTEGER PROGRAMMING FORMULATION OF THE QUADRATIC ASSIGNMENT PROBLEM WITH RECTANGULAR DISTANCES

Suppose n locations are given as points in the first quadrant of a two-dimensional plane (given points can always be represented on the first quadrant by a translation of axes), and suppose the flow between each pair of facilities is given. If one uses rectangular distances, which sometimes are more appropriate in industrial settings than straight-line distances [6, 14], then a binary mixed integer program to solve the quadratic assignment problem is as follows:

Minimize $$\sum_{i=1}^{n-1} \sum_{j=i+1}^{n} w_{ij}(R_{ij}+L_{ij}+A_{ij}+B_{ij})$$

subject to

$$\left.\begin{aligned} R_{ij}-L_{ij}&=x_i-x_j \\ A_{ij}-B_{ij}&=y_i-y_j \end{aligned}\right\} \begin{aligned} &i=1,\ldots,n-1 \\ &j=i+1 \end{aligned}$$

(1) $$x_i+y_i=\sum_{k=1}^{n} s_k\alpha_{ik}, \qquad i=1,\ldots,n$$

(2) $$x_i-y_i=\sum_{k=1}^{n} d_k\alpha_{ik}, \qquad i=1,\ldots,n$$

$$\sum_{k=1}^{n}\alpha_{ik}=1, \qquad i=1,\ldots,n$$

$$\sum_{i=1}^{n}\alpha_{ik}=1, \qquad k=1,\ldots,n$$

$$\alpha_{ik}=0 \text{ or } 1, \qquad i,k=1,\ldots,n$$

$$R_{ij},\ L_{ij},\ A_{ij},\ B_{ij},\ x_1,\ldots,x_n,\ y_1,\ y_2,\ \ldots,\ y_n\geq 0.$$

Where

n=number of facilities and the number of locations,
w_{ij}=nonnegative weight or flow between facility i and facility j,
R_{ij}=horizontal distance between facility i and facility j if facility i is to the right of facility j,
=O, otherwise,
L_{ij}=horizontal distance between facility i and facility j if facility i is to the left of facility
=O, otherwise,
A_{ij}=vertical distance between facility i and facility j if facility i is above facility j,
=O, otherwise,
B_{ij}=vertical distance between facility i and facility j if facility i is below facility j,
=O, otherwise
(x_i, y_i)=location of facility i, $i=1, 2, \ldots, n$,
s_k=sum of coordinates of location k, $k=1, 2, \ldots, n$,
d_k=difference of coordinates of location k, value of the first coordinate minus value of the second coordinate.

To illustrate constraints (1) and (2) of above, consider the point (2, 3). Then $x_i+y_i=5$ specifies a line with slope -1 which passes through (2, 3); $x_i-y_i=-1$ specifies a line with slope $+1$ which also passes through (2, 3). $x_i+y_i=5$ and $x_i-y_i=-1$ together specify that facility i is located at location (2, 3), for the point (2, 3) is the unique intersection of these two lines. Similarly, any point in a two-dimensional Euclidean space has a unique sum and difference of coordinates.

$\sum_{k=1}^{n}\alpha_{ik}=1$ specifies that facility i can only be located on one of the given n locations. $\sum_{i=1}^{n}\alpha_{ik}=1$ specifies that each location is occupied. The formulation involves n^2 integer variables all of which are binary, and has n^2+3n constraints excluding the nonnegativity conditions.

The formulation can be extended to cover the more general case where "site costs" c_{ij} are given. The only modification is to add the following to the objective function:

$$\sum_{i=1}^{n}\sum_{k=1}^{n} c_{ik}\alpha_{ik},$$

ere c_{ik} is the fixed cost of locating facility i at location k. It is important to note that this extended f mulation does not involve additional variables, because the number of integer variables may be t most important factor affecting the amount of computation for computer codes using branch- d-bound algorithms [2]; the number of constraints remains the same.

The integer programming formulation also can accommodate linear locational constraints. For i tance, to fix the location of department i, (x_i, y_i), we equate (x_i, y_i) to the coordinates of the cation where we want to put the facility. Thus we can take care of situations like "the desire of executive to be placed in a particular location" [10].

3 COMPUTATIONAL EXPERIENCE

Ising the IBM MIP code [7], the formulation was applied to four numerical problems. The data f problem 1 is given in Table 1. Problems 2, 3, and 4 are the 5-department, 8-department, and l department problems, respectively, of Nugent, *et al.* [10]. Some computational results are pre- s ted in Table 2.

TABLE 1(a). *Available locations*

Location name	Where situated
A	(0, 0)
B	(3, 3.7)
C	(5, 1)

TABLE 1(b). *Inter-facility flow*

	Facility 1	2	3
1	—	5	6
Facility 2	—	—	7
3	—	—	—

TABLE 1(c). *"Site Cost" data*

	Location A	B	C
1	15	11	19
Facility 2	14	17	12
3	18	13	16

TABLE 2. *Computational results for problems 1, 2, 3, and 4*

	Problem 1	Problem 2	Problem 3	Problem 4
BM computer used	360/65	360/65	370/158	370/158
erations	58	498	7738	2067
PU time	0.121 min.	0.544 min.	9.64 min.	4.60 min.
ptimality obtained	Yes	Yes	Yes	No
est solution obtained:	f1*:(0, 0) f2:(5, 1) f3:(3, 3.7) cost=143.10	f1:(1, 1) f2:(1, 0) f3:(2, 0) f4:(0, 1) f5:(0, 0) cost=25	f1=(1, 0) f2=(0, 0) f3=(0, 1) f4=(2, 0) f5=(3, 0) f6=(3, 1) f7=(2, 1) f8=(1, 1) cost=107	f1=(2, 1) f2=(3, 2) f3=(1, 1) f4=(1, 2) f5=(2, 2) f6=(3, 1) f7=(0, 1) f8=(0, 2) f9=(1, 0) f10=(3, 0) f11=(2, 0) f12=(0, 0) cost=345

*fi stands for facility i.

For problem 3, the obtained functional value of 107 is known to be optimal. For problem 4 the obtained functional value of 345 is known to be not attractive [10]. In a subsequent run, n solution of 300 or less was obtained after 7686 iterations in 20 CPU minutes. It was then decided t devise a procedure to get good solutions for these larger problems.

The procedure requires an examination of given data. First, fix some facilities at locations s that most large flows have only short distances to travel. Solve this restricted problem. Then, chang the configuration of the fixed facilities and re-solve. Repeat this procedure several times.

To illustrate, we again use problem 4, for there is considerable history behind it [10]. First find the two facilities with the largest sum of flows, and locate them at two central locations. Sinc facility 11 has the largest sum of flows and facility 8 and facility 9 tie for the second largest sum o flows, fix facility 11 at (1, 1) and facility 8 at (2, 1) for one possibility; fix facility 11 at (1, 1 and facility 9 at (2, 1) for another possibility. The third facility to be located is one which has th largest sum of flows to the two already fixed facilities; fix this facility at a location so that the trans portation cost between it and the two fixed facilities is smallest. The fourth facility is chosen an located by treating the first three facilities as fixed. Similarly the fifth and the sixth facilities ar chosen and located. There was no need to fix additional facilities because the integer programmin code used can readily handle six free facilities. Table 3 shows five such configurations of fixed facil ities; it also shows some computational results. It is noted that the functional value of 289 of Run 2 i probably optimal [10].

TABLE 3. *Fixed Configurations and Corresponding Computationals Results*

	Run 1	Run 2	Run 3	Run 4	Run 5
Fixed facilities in sequence of fixing:	f11: (1, 1) f8: (2, 1) f4: (2, 0) f7: (3, 1) f9: (1, 0) f12: (3, 0)	f11=(1, 1) f8=(2, 1) f4=(3, 1) f7=(2, 0) f9=(1, 0) f12=(3, 0)	f11=(1, 1) f9=(2, 1) f12=(2, 0) f7=(3, 1) f8=(3, 0) f4=(1, 0)	f11=(1, 1) f9=(2, 1) f12=(2, 0) f7=(3, 1) f8=(3, 2) f4=(2, 2)	f11=(1, 1) f9=(2, 1) f12=(3, 1) f7=(2, 0) f8=(1, 0) f4=(3, 0)
IBM Computer used	370/158	370/158	370/158	370/158	370/158
iterations	5136	4064	4438	5099	4216
CPU time for run	11.67 min.	9.20 min.	10 min.	11.79 min.	9.46 min.
optimality proved?	Yes	Yes	No*	Yes	Yes
Best solution obtained:	f1=(0, 1) f2=(0, 2) f3=(0, 0) f5=(2, 2) f6=(3, 2) f10=(1, 2) cost=306	f1=(0, 1) f2=(0, 2) f3=(0, 0) f5=(3, 2) f6=(2, 2) f10=(1, 2) cost=289	f1=(3, 2) f2=(1, 2) f3=(2, 2) f5=(0, 0) f6=(0, 1) f10=(0, 2) cost=305	f1=(1, 2) f2=(1, 0) f3=(3, 0) f5=(0, 2) f6=(0, 1) f10=(0, 0) cost=305	f1=(0, 0) f2=(0, 2) f3=(0, 1) f5=(3, 2) f6=(2, 2) f10=(1, 2) cost=296
CPU time when this solution was obtained:	5.69 min.	2.72 min.	9.49 min.	6.03 min.	6.58 min.

*The MIP code indicated a lower bound on the optimal solution of 293.5

4. CONCLUDING REMARKS

The formulation presented here for solving the quadratic assignment problem with rectangular distances in a two-dimensional space has n^2 binary variables and (n^2+3n) constraints. It involves ordered sets which can be processed by some integer programming codes [8]. The integer programming approach, although restricted to relatively small problems, has the advantage of not requiring a specialized quadratic assignment code, will accommodate linear constraints, and guarantees optimality given sufficient computing time. For problems of size larger than 8 or 9 facilities, very good solutions may be obtained by considering several runs, each of which has a subset of facilities fixed. With the development of more efficient integer programming codes the attractiveness of the method will increase.

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MULTIVARIATE TESTS OF FIT USING ASYMPTOTICALLY SUFFICIENT GROUPING

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ABSTRACT

A sample is taken from a continuous multivariate distribution. The problem is to test the hypothesis that the unknown joint cumulative distribution function is equal to a completely specified function. The observed data are transformed so that the hypothesis being tested is that the distribution is uniform over a unit hypercube. If only neighboring alternatives are considered, it is shown that the numbers of observations falling in a gradually increasing number of subcubes are asymptotically sufficient. It is shown that for all asymptotic probability calculations, we can assume that the joint distribution of the numbers of observations can be considered to be the distribution of slightly rounded off normal random variables. Tests based on these facts are constructed.

N'RODUCTION

For each positive integer n, $Y_1(n), \ldots, Y_n(n)$ are independent and identically distributed -mensional random variables, the joint cumulative distribution function for the components of Y_i) being $F_n(x_1, \ldots, x_s)$. $F_n(x_1, \ldots, x_s)$ is absolutely continuous, but is otherwise unknown. T problem is to test the hypothesis that $F_n(x_1, \ldots, x_s)$ is equal to $G_n(x_1, \ldots, x_s)$ for all $x_1, \ldots, x_s$, v re $G_n(x_1, \ldots, x_s)$ is a completely specified absolutely continuous cumulative distribution uction. The reason we allow these distributions to depend on n is that we want to allow F_n and G_n o approach each other as n increases.

In the case $s=1$, many tests have been proposed for this problem, including tests of h Kolmogorov-Smirnov and Cramér-Von Mises types. A well known example of Simpson [2] h vs that the testing problem for $s>1$ is fundamentally different from the problem for $s=1$. The mber of tests proposed and analyzed for the case $s>1$ is much smaller than for the case $s=1$.)r general type of test that is familiar for all s is the chi-square test based on grouped data. If the mber of groups is fixed, the test is not consistent against certain alternatives. In this paper, we t y tests based on grouped data, where the number of groups increases as n increases. In deciding n ow the number of groups should be made to depend on the sample size, there are two con- ing criteria: (1) If the number of groups is relatively small, a lot of information is lost in the r ping; (2) if the number of groups is relatively large, the asymptotic distribution theory needed

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to analyze the properties of the tests becomes more complicated. In this paper, we choose t number of groups to achieve two goals: (1) For a class of alternatives close to the distribution bei tested, asymptotically (as $n\to\infty$) there is no loss of information caused by the grouping, so that may consider the grouping to be asymptotically sufficient; (2) for a wider class of alternativ for all probability calculations, the numbers of observations falling in the various groups can considered to be slightly rounded-off normal random variables.

The paper is organized as follows. In Section 2, we define a class of alternatives for whi asymptotically there is no loss of information caused by the grouping. In Section 3, we define t wider class of alternatives under which we can assume the numbers of observations falling in t various groups are rounded-off normal variables. In Section 4 we give the proof of the statemer in Section 2. Section 5 contains a detailed example. Section 6 contains suggestions for furth research.

2. ASYMPTOTIC SUFFICIENCY

By applying the transformation described by Rosenblatt [1], based on G_n, to each $Y_i(n)$, get s-dimensional independent and identically distributed random variables $X_1(n), \ldots, X_n($ whose common distribution assigns probability one to the s-dimensional unit cube C_s: $(0\leqq x_i\leqq$ $i=1, \ldots, s$), and whose common distribution is uniform over C_s if, and only if, the hypothe described above is true. From now on we use $X_1(n), \ldots, X_n(n)$ instead of $Y_1(n), \ldots, Y_n(n)$, that we are testing the hypothesis that the common distribution of $X_1(n), \ldots, X_n(n)$ is unifo over C_s.

For the rest of this section, we limit attention to alternatives under which $X_i(n)$ has probabil density function of the form $1+r_n(x_1, \ldots, x_s)$ if $(x_1, \ldots, x_s)$ is in C_s, zero if $(x_1, \ldots, x_s)$ is n in C_s, where

$$\int\cdots\int_{C_s} r_n(x_1, \ldots, x_s)dx_1\cdots dx_s=0, \frac{\partial r_n(x_1, \ldots, x_s)}{\partial x_i} \text{ and } \frac{\partial^2 r_n(x_1, \ldots, x_s)}{\partial x_i\partial x_j}$$

exist for all $(x_1, \ldots, x_s)$ in the interior of C_s and all $i, j=1, \ldots, s$, and

$$|r_n(x_1, \ldots, x_s)|<B(n), \left|\frac{\partial r_n(x_1, \ldots, x_s)}{\partial x_i}\right|<B(n), \left|\frac{\partial^2 r_n(x_1, \ldots, x_s)}{\partial x_i\partial x_j}\right|<B(n)$$

for all $(x_1, \ldots, x_s)$ in the interior of C_s and $i, j=1, \ldots, s$, with $\{B(n)\}$ a nonrandom sequen satisfying

$$\lim_{n\to\infty} n[B(n)]^3=0.$$

Our hypothesis is that $r_n(x_1, \ldots, x_s)=0$ for all $(x_1, \ldots, x_s)$ in C_s. We discuss these assumptio below.

For each n, we break C_s into $k(n)$ equal subcubes, so that each subcube has sides of leng $[k(n)]^{-1/s}=L(n)$, say, where

$$\lim_{n\to\infty} n^{-1}[k(n)]^3=0$$

and

$$\lim_{n\to\infty} \{n[B(n)]^2[k(n)]^{-1/s}\}=0.$$

Let $S_n(1), \ldots, S_n(k(n))$ denote the subcubes listed in some definite order, let $z_n(\beta)$ denote the center of subcube $S_n(\beta)$, and let $N_n(\beta)$ denote the number of $\{X_i(n)\}$ in subcube $S_n(\beta)$. We can ignore the possibility that an $X_i(n)$ will fall on the boundary of a subcube. For each β, let $\{Z(1, \beta, n), \ldots, Z(N_n(\beta), \beta, n)\}$ denote the $N_n(\beta)$ vectors among the $\{X_i(n)\}$ which fall in $S_n(\beta)$. Assume the Z's are listed in the same order as they appeared among the $\{X_i(n)\}$.

The joint conditional distribution of $\{Z(j, \beta, n); j=1, \ldots, N_n(\beta)\}$, given $N_n(\beta)$, is that of $N_n(\beta)$ independent and identically distributed random variables, each with density function equal to

$$\frac{1+r_n(x_1, \ldots, x_s)}{\int_{S_n(\beta)} \cdots \int [1+r_n(x_1, \ldots, x_s)]dx_1 \ldots dx_s}$$

if $(x_1, \ldots, x_s)$ is in $S_n(\beta)$, and equal to zero if $(x_1, \ldots, x_s)$ is not in $S_n(\beta)$. Given $\{N_n(\beta); \beta=1, \ldots, k(n)\}$, $Z(i, \alpha, n)$, $Z(j, \beta, n)$ are independent if $\alpha \neq \beta$. The joint distribution of $\{N_n(\beta)\}$ is multinomial, with parameters n,

$$\left\{\int_{S_n(\beta)} \cdots \int [1+r_n(x_1, \ldots, x_s)]dx_1 \ldots dx_s\right\}.$$

Let $p_n(\ )$ denote the joint density for $\{N_n(\beta), Z(j, \beta, n); \beta=1, \ldots, k(n), j=1, \ldots, N_n(\beta)\}$. For any event E, let $P_n(E)$ denote the probability of the event E when $p_n(\ )$ is the joint density for $\{N_n(\beta), Z(j, \beta, n)\}$.

Now we construct an artificial joint density for $\{N_n(\beta), Z(j, \beta, n)\}$, which we denote by $\overline{p}_n(\ )$, as follows. The joint distribution of $\{N_n(\beta)\}$ remains the same as above, but the joint conditional distribution of $\{Z(j, \beta, n); j=1, \ldots, N_n(\beta)\}$, given $N_n(\beta)$, is that of $N_n(\beta)$ independent random variables, each with a uniform distribution over $S_n(\beta)$. For any event E, let $\overline{P}_n(E)$ denote the probability of the event E when $\overline{p}_n(\ )$ is the joint density for $\{N_n(\beta), Z(j, \beta, n)\}$.

In Section 4, we prove the following

THEOREM: For each n, let R_n be an arbitrary measurable region in the $(k(n)+n)$-dimensional space of $\{N_n(\beta), Z(j, \beta, n)\}$, and let E_n denote the event that the $(k(n)+n)$-dimensional point $\{N_n(\beta), Z(j, \beta, n)\}$ falls in R_n. Then

$$\lim_{n\to\infty} |P_n(E_n) - \overline{P}_n(E_n)| = 0.$$

It is a consequence of this theorem that $\{N_n(\beta); \beta=1, \ldots, k(n)\}$ are asymptotically sufficient for our hypothesis testing problem against the alternatives described above. This can be seen as follows. Given $\{N_n(\beta)\}$, we could generate n random variables whose joint conditional distribution is the same as that given by $\overline{p}_n(\ )$. We could do this because under $\overline{p}_n(\ )$, the joint conditional distribution of $\{Z(j, \beta, n)\}$ given $\{N_n(\beta)\}$ is known, and does not depend on the unknown alternative to the hypothesis. But the theorem shows that asymptotically the probability assigned to any sequence of regions by $\overline{p}_n(\ )$ is the same as that assigned by $p_n(\ )$. Therefore a statistician who knows only $\{N_n(\beta)\}$ is asymptotically just as well off as a statistician who knows $\{N_n(\beta), Z(j, \beta, n)\}$. In this sense, $\{N_n(\beta)\}$ are asymptotically sufficient.

We conclude this section by discussing the reasonableness of the assumptions we have made about the alternatives allowed. First, suppose we look at alternative densities equal to $1+n^{-\delta}r(x_1,$

. . ., x_s) if (x_1, . . ., x_s) is in C_s, and equal to zero if (x_1, . . ., x_s) is not in C_s, where δ is a fixe positive value, and

$$\int \cdots \int_{C_s} r(x_1, \ldots, x_s) dx_1 \ldots dx_s = 0.$$

If $\delta > \frac{1}{2}$, the asymptotic power of any sequence of tests is easily seen to be equal to the asymp totic level of significance, so such alternatives are too close to the null hypothesis. If $\delta < \frac{1}{2}$, it i easy to construct tests whose asymptotic power is 1, so such alternatives are too far from the nul hypothesis. Thus $\delta = \frac{1}{2}$ gives the alternatives of interest, and our class of alternatives allowe above includes the case $\delta = \frac{1}{2}$.

The discussion in the preceding paragraph assumes we have already transformed to the uni square C_s. Now let us look at a situation before we transform to C_s. Suppose $s=2$, and that for each n, the density function $g_n(x_1, x_2)$ being tested is

$$\frac{1}{2\pi\sqrt{1-\rho^2}} \exp\left\{-\frac{1}{2(1-\rho^2)}[x_1^2+x_2^2-2\rho x_1 x_2]\right\}$$

where ρ is in the open interval $(-1, 1)$. Suppose the alternative is

$$f_n(x_1,x_2)=\frac{1}{2\pi\sqrt{(1-\rho_n^2)}} \exp\left\{-\frac{1}{2(1-\rho_n^2)}[x_1^2+x_2^2-2\rho_n x_1 x_2)]\right\},$$

where

$$\rho_n = \rho + \frac{\Delta}{\sqrt{n}}$$

for some fixed nonzero Δ. The transformation used to get to C_2 preserves the ratios

$$\frac{f_n(x_1,x_2)}{g_n(x_1,x_2)}.$$

These ratios become unbounded for some (x_1, x_2), so when the transformation to C_2 is made our assumptions about the alternative made above are violated. The following device can be used to make our results applicable in this case. Define $\bar{f}_n(x_1, x_2)$ as the following density function $\bar{f}_n(x_1, x_2) = D_1(n) f_n(x_1, x_2)$ if $|x_1|+|x_2|<n^{1/24}$, $\bar{f}_n(x_1, x_2)=0$ if $|x_1|+|x_2|\geqq n^{1/24}$, where $D_1(n)$ is a normalizing constant. Define $\bar{g}_n(x_1, x_2)$ as the following density function: $\bar{g}_n(x_1, x_2)=D_2(n)g_n(x_1, x_2)$ if $|x_1|+|x_2|<n^{1/24}$, $\bar{g}_n(x_1, x_2)=0$ if $|x_1|+|x_2|\geqq n^{1/24}$. It is easily verified that if we are testing that $\bar{g}_n(x_1, x_2)$ is the common *pdf* against the alternative $\bar{f}_n(x_1, x_2)$, then after the transformation to C_2 our assumptions above are satisfied. It is also easily verified that all asymptotic probabilities are the same under $\bar{f}_n$ as under f_n, and are the same under $\bar{g}_n$ as under g_n. Thus in testing g_n against f_n, we may as well assume we are testing $\bar{g}_n$ against $\bar{f}_n$, and asymptotically the problems are equivalent. This device can be used in many cases to make our results applicable.

3 THE ASYMPTOTIC DISTRIBUTION OF $\{N_n(\beta)\}$

In this section, we relax the assumptions about $r_n(x_1, \ldots, x_s)$ to the following:

$$\sup_{C_s} |r_n(x_1, \ldots, x_s)| < \beta(n).$$

The joint distribution of $\{N_n(\beta)\}$ is multinomial, with parameters

$$n, \{\int_{S_n(\beta)} \cdots \int [1+r_n(x_1, \ldots, x_s)]dx_1 \cdots dx_s\}.$$

Define $T_n(\beta)$ as

$$[n/k(n)]^{-1/2}[N_n(\beta) - n\int_{S_n(\beta)} \cdots \int [1+r_n(x_1, \ldots, x_s)]dx_1 \cdots dx_s].$$

Let $q_n(t_n(1), \ldots, t_n(k(n)))$ denote $P[T_n(\beta)=t_n(\beta); \beta=1, \ldots, k(n)]$.

Suppose random variables $\{Z_n(1), \ldots, Z_n(k(n)-1)\}$ have the following joint normal probability density function:

$$[1/2\pi]^{[k(n)-1]/2}\sqrt{k(n)} \exp\left\{-\frac{1}{2}\sum_{j=1}^{k(n)} z_i^2\right\}$$

where $z_{k(n)}$ is defined as

$$-\sum_{i=1}^{k(n)-1} z_i.$$

Define the random variables $\overline{Z}_n(1), \ldots, \overline{Z}_n(k(n))$ as the following functions of $Z_n(1), \ldots, Z_n(k(n)-1)$. For $i=1, \ldots, k(n)-1$, $\overline{Z}_n(i)$ is the closest value to $Z_n(i)$ which makes

$$n\int_{S_n(\beta)} \cdots \int [1+r_n(x_1, \ldots, x_s)]dx_1 \cdots dx_s + \sqrt{n/k(n)}\,\overline{Z}_n(i)$$

an integer (positive, negative, or zero). If there are two possible values for $\overline{Z}_n(i)$, use the smaller one. $\overline{Z}_n(k(n))$ is defined as

$$-\sum_{i=1}^{k(n)-1} \overline{Z}_n(i).$$

Let $\bar{q}_n(\bar{z}_n(1), \ldots, \bar{z}_n(k(n))$ denote

$$P[\overline{Z}_n(i)=\bar{z}_n(i); i=1, \ldots, k(n)].$$

For any region S_n in $k(n)$-dimensional space, let $P_{q_n}(S_n)$ and $P_{\bar{q}_n}(S_n)$ denote the probabilities assigned to S_n by q_n and $\bar{q}_n$ respectively. Then we have the following:

THEOREM:

$$\lim_{n\to\infty} |P_{q_n}(S_n) - P_{\bar{q}_n}(S_n)| = 0$$

for any sequence $\{S_n\}$, where S_n is an arbitrary region in $k(n)$-dimensional space.

This theorem is a special case of a theorem proved in [5].

We note that the same device used at the end of Section 2 can be applied to make the theorem of this section applicable to a wide class of pairs $\{f_n, g_n\}$.

4. PROOF OF THEOREM OF SECTION 2

By the argument given in [3], the theorem will be proved if we can show that

$$(4.1)\quad \log \frac{p_n(\{N_n(\beta), Z(j,\beta,n)\})}{\bar{p}_n(\{N_n(\beta), Z(j,\beta,n)\})} \equiv \sum_{\beta=1}^{k(n)} \sum_{j=1}^{N_n(\beta)} \left\{ \begin{array}{l} \log [1+r_n(Z(j,\beta,n))] \\ -\log [1+k(n) \int_{S_n(\beta)} \cdots \int r_n(x_1, \ldots, x_s) dx_1 \cdots dx_s] \end{array} \right\}$$

converges stochastically to zero as n increases, assuming the joint density for $\{N_n(\beta), Z(j,\beta,n)\}$ is $\bar{p}_n(\ \ )$.

Using the expansion $\log (1+y) = y - \frac{y^2}{2} + \frac{y^3}{3} - \ \ldots$ for $|y|<1$, the assumptions about $r_n(\ \ )$, $k(n)$, and $B(n)$, and the fact that the volume of $S_n(\beta)$ is $\frac{1}{k(n)}$ it is easily seen that (4.1) can be written as the sum of the following three expressions:

$$(4.2)\quad \sum_{\beta=1}^{k(n)} \sum_{j=1}^{N_n(\beta)} \{r_n(Z(j,\beta,n)) - k(n) \int_{S_n(\beta)} \cdots \int r_n(x_1, \ldots, x_s) dx_1 \cdots dx_s\}$$

$$(4.3)\quad \frac{1}{2} \sum_{\beta=1}^{k(n)} \sum_{j=1}^{N_n(\beta)} \{[k(n) \int_{S_n(\beta)} \cdots \int r_n(x_1, \ldots, x_s) dx_1 \cdots dx_s]^2 - [r_n Z(j,\beta,n)]^2\}$$

$$(4.4)\quad \Delta(n)$$

where $\Delta(n)$ converges stochastically to zero as n increases.

For any point $(x_1, \ldots, x_s)$ in $S_n(\beta)$, we can write

$$r_n(x_1, \ldots, x_s) = r_n(z_n(\beta)) + \sum_{i=1}^{s} \frac{\partial r_n(x)}{\partial x_i}\Big]_{x=z_n(\beta)} (x_i - z_n^{(i)}(\beta)) + \frac{1}{2} \sum_{j=1}^{s} \sum_{i=1}^{s} \frac{\partial^2 r_n(x)}{\partial x_i \partial x_j}\Big]_{x=x^*} (x_i - z_n^{(i)}(\beta))(x_j - z_n^{(j)}(\beta))$$

where $z_n^{(i)}(\beta)$ is the ith coordinate of $z_n(\beta)$, and x^* is some point in $S_n(\beta)$. From the expansion it is easily found that we can write

$$\int_{S_n(\beta)} \cdots \int r_n(x_1, \ldots, x_s) dx_1 \cdots dx_s$$

as

$$\frac{r_n(z_n(\beta))}{k(n)} + \theta_n(\beta) B(n) (L(n))^{s+2},$$

where $|\theta_n(\beta)| \leqq 1$. It follows that we can write

$$[k(n) \int_{S_n(\beta)} \int r_n(x_1, \ldots, x_s) dx_1 \cdots dx_s]^2$$

as

$$[r_n(z_n(\beta))]^2 + 2\bar{\theta}_n(\beta) [B(n)]^2 [k(n)]^{-2/s},$$

where $|\bar{\theta}_n(\beta)| \leqq 1$. From the expansion, it also follows that we can write $[r_n(Z_n(j,\beta,n))]^2$ as $[r_n(z_n(\beta))]^2 + 2\theta^*_n(\beta)(B(n))^2\ [k(n)]^{-1/s}$, where $|\theta^*_n(\beta)| \leqq 1$. From these facts, and the assumptions about $k(n)$ and $B(n)$, it follows that the expression (4.3) converges stochastically to zero as n increases.

Next we examine the expression (4.2). Recalling that we are assuming that $\bar{p}_n(\ )$ is the density, the conditional mean and conditional variance of $r_n(Z(j, \beta, n))$, given $N_n(\beta)$, are respectively

$$k(n)\int_{S_n(\beta)} \cdots \int r_n(x_1, \ldots, x_s)dx_1 \ldots dx_s,\ k(n)\int_{S_n(\beta)} \cdots \int [r_n(x_1, \ldots, x_s)]^2 dx_1 \cdots dx_s$$
$$-[k(n)\int_{S_n(\beta)} \cdots \int r_n(x_1, \ldots, x_s)dx_1 \cdots dx_s]^2.$$

Using the expansions above, this conditional variance is not greater than $4[B(n)]^2(k(n))^{-1/s}$. It follows that the conditional expected value of the expression (4.2) is zero, and the conditional variance converges stochastically to zero as n increases. This implies that the expression (4.2) converges stochastically to zero as n increases, and completes the proof of the theorem.

5. APPLICATION TO TESTING FIT

The results above are analogous to the results in [4], which discussed the case $s=1$. In [4], a certain set of order statistics was shown to be asymptotically sufficient and asymptotically normal. In the present paper, $\{N_n(\beta)\}$ have been shown to be asymptotically sufficient and asymptotically equivalent to slightly rounded-off normal variables. The result of Simpson [2] cited above shows that we cannot expect an exact analogy between the cases $s=1$ and $s>1$, but we will develop a test of fit for our present case which is analogous to the test given in [4]. The following discussion is based on the discussion in Section 4 of [4].

First, suppose we are testing the hypothesis $r_n(x_1, \ldots, x_s)=0$ against one specific sequence of alternatives $\{\bar{r}_n(x_1, \ldots, x_s)\}$, assumed to satisfy the condition of Section 2. Let $\Phi(t)$ denote

$$\int_t^\infty (2\pi)^{-1/2} \exp\left(-\frac{y^2}{2}\right) dy,$$

and define z_α by $\Phi(z_\alpha)=\alpha$. As in [4], the maximum possible asymptotic power of a test of level of significance α is

$$\Phi[z_\alpha - \{\int_{C_s} \cdots \int \bar{r}_n^2(x_1, \ldots, x_s)dx_1 \cdots dx_s\}^{1/2}],$$

so that a natural measure of distance between the uniform density over C_s and the density $1+r_n(x_1, \ldots, x_s)$ is

$$\int_{C_s} \cdots \int r_n^2(x_1, \ldots, x_s)dx_1 \cdots dx_s.$$

Assuming that $\{r_n(x_1, \ldots, x_s)\}$ satisfies the condition of Section 2, we expand $r_n(x_1, \ldots, x_s)$ in a Fourier cosine series over C_s:

$$r_n(x_1, \ldots, x_s)=\sum_{j_1=0}^{\infty} \cdots \sum_{j_s=0}^{\infty} A_n(j_1, \ldots, j_s)2^{D(j_1, \ldots, j_s)} \cos j_1\pi x_1 \cdots \cos j_s\pi x_s,$$

where $D(j_1, \ldots, j_s)$ is equal to one-half of the number of positive values among $j_1, \ldots, j_s$, and $A(j_1, \ldots, j_s)$ is equal to

$$\int_{C_s} \cdots \int r_n(x_1, \ldots, x_s)\, 2^{D(j_1, \ldots, j_s)} \cos j_1\pi x_1 \cdots \cos j_s\pi x_s dx_1 \cdots dx_s.$$

$$A_n(0, \ldots, 0)=0.$$

The center of $S_n(\beta)$ is $z_n(\beta)$. Write the coordinates of $z_n(\beta)$ as $(z_n^{(1)}(\beta), \ldots, z_n^{(s)}(\beta))$. For any given nonnegative integers $j_1, \ldots, j_s$, not all zero, define $\Gamma(j_1, \ldots, j_s; z_n(\beta))$ as $2^{D(j_1,\ldots,j_s)} \cos j_1\pi z_n^{(1)}(\beta) \ldots \cos j_s\pi z_n^{(s)}(\beta)$. Define $\bar{A}_n(j_1, \ldots, j_s)$ as

$$n^{-1/2}\left\{\sum_{\beta=1}^{k(n)} N_n(\beta)\Gamma(j_1, \ldots, j_s; z_n(\beta)) - \frac{n}{k(n)}\sum_{\beta=1}^{k(n)}\Gamma(j_1, \ldots, j_s; z_n(\beta))\right\}.$$

Note that $\bar{A}_n(j_1, \ldots, j_s)$ is observable. We have the following:

THEOREM: Let $\{j_1^{(i)}, \ldots, j_s^{(i)}; i=1, \ldots, m\}$ be m different sets of s nonnegative integers each set containing at least one positive value. Then the asymptotic (as $n\to\infty$) joint distribution of $\{\bar{A}_n(j_1^{(i)}, \ldots, j_s^{(i)}); i=1, \ldots, m\}$ is that of m independent normal random variables, with variances one and expected values

$$\{\sqrt{n}A_n(j_1^{(i)}, \ldots, j_s^{(i)}); i=1, \ldots, m\}.$$

PROOF: Write $N_n(\beta)$ in terms of $T_n(\beta)$ in the definition of $\bar{A}_n(j_1, \ldots, j_s)$, and then replace $T_n(\beta)$ by $\bar{Z}_n(\beta)$. The theorem of Section 3 tells us that this cannot affect the asymptotic distribution. For $\beta=1, \ldots, k(n)-1$, we have

$$\bar{Z}_n(i)=Z_n(i)+\frac{\theta_n(i)}{2}\sqrt{\frac{k(n)}{n}},$$

where $|\theta_n(i)|\leqq 1$. Defining $Z_n(k(n))$ as

$$-\sum_{i=1}^{k(n)-1} Z_n(i),$$

we have

$$\bar{Z}_n(k(n))=Z_n(k(n))+\frac{\theta_n(k(n))}{2}\sqrt{\frac{(k(n))^3}{n}}.$$

The theorem follows from a simple calculation, using the easily verified facts that

$$\text{Variance } (Z_n(i))=\frac{k(n)-1}{k(n)} \ (i=1, \ldots, k(n)), \ \text{Cov } (Z_n(i), Z_n(j))=-\frac{1}{k(n)} \text{ for } i\neq j.$$

The theorem just proved shows that

$$\sum_{i=1}^{m} \bar{A}_n^2(j_1^{(i)}, \ldots, j_s^{(i)})$$

has asymptotically a noncentral chi-square distribution with m degrees of freedom and noncentrality parameter

$$n\sum_{i=1}^{m} A_n^2(j_1^{(i)}, \ldots, j_s^{(i)}).$$

We can use the statistic

$$\sum_{i=1}^{m} \bar{A}_n^2(j_1^{(i)}, \ldots, j_s^{(i)})$$

to test our hypothesis, which is equivalent to the hypothesis that the noncentrality parameter is zero. Just which sets $\{j_1, \ldots, j_s\}$ should be included in our test statistic depends on the alternatives of interest. A suggestion about the choice is given in [4].

To illustrate this test procedure, suppose $s=2$ and we want to test the hypothesis that we are sampling from a bivariate normal distribution with zero means, unit variances, and correlation coefficient equal to 0.4. A Monte Carlo sample of 100 independent pairs was taken from a bivariate normal population with zero means, unit variances, and correlation coefficient equal to 0.6. Let (T_t, U_t) denote the ith pair observed ($i=1, \ldots, 100$). Recalling our definition of Φ, the transformation from [1] that we use replaces T_t by $1-\Phi(T_t)$ and U_t by

$$1-\Phi\left(\frac{U_t-0.4T_t}{\sqrt{1-0.16}}\right).$$

Note that we could have interchanged the roles of T_t and U_t: the fact that there is more than one possible transformation for $s>1$ makes the cases $s=1$ and $s>1$ fundamentally different. $k(n)$ was set equal to 9. $z_n(\beta)$ and $N_n(\beta)$ were as follows:

$$z_n(1)=\left(\frac{1}{6}, \frac{1}{6}\right), \qquad N_n(1)=9$$

$$z_n(2)=\left(\frac{1}{2}, \frac{1}{6}\right), \qquad N_n(2)=4$$

$$z_n(3)=\left(\frac{5}{6}, \frac{1}{6}\right), \qquad N_n(3)=8$$

$$z_n(4)=\left(\frac{1}{6}, \frac{1}{2}\right), \qquad N_n(4)=17$$

$$z_n(5)=\left(\frac{1}{2}, \frac{1}{2}\right), \qquad N_n(5)=18$$

$$z_n(6)=\left(\frac{5}{6}, \frac{1}{2}\right), \qquad N_n(6)=16$$

$$z_n(7)=\left(\frac{1}{6}, \frac{5}{6}\right), \qquad N_n(7)=6$$

$$z_n(8)=\left(\frac{1}{2}, \frac{5}{6}\right), \qquad N_n(8)=12$$

$$z_n(9)=\left(\frac{5}{6}, \frac{5}{6}\right), \qquad N_n(9)=10.$$

Which sets (j_1, j_2) are of interest here? If we were interested in alternatives which differ from the null hypothesis only in having the correlation coefficient different from 0.4 (as is the case here), then we should emphasize the second coordinate, because the transformation above uses the correlation coefficient only for transforming the second coordinate. Suppose we use the following pairs (j_1, j_2): (0, 1), (1, 0), (1, 1), (0, 2). We then calculate

$$\overline{A}_n(0, 1)=-\frac{7}{10}\sqrt{\frac{3}{2}}, \qquad \overline{A}_n(1, 0)=\frac{1}{5}\sqrt{\frac{3}{2}}, \qquad \overline{A}_n(1, 1)=\frac{3}{4}, \qquad \overline{A}_n(0, 2)=-\sqrt{2}(2.65).$$

$$\sum_{i=1}^{4} \overline{A}_n{}^2(j_1{}^{(i)}, j_2{}^{(i)})=15.40.$$

Comparing this to chi-square critical values for 4 degrees of freedom, it falls between 0.01 and 0.01.

6. CONCLUDING REMARKS

The test based on

$$\sum_{i=1}^{m} A_n^{-2}(j_1^{(i)}, \ldots, j_s^{(i)})$$

is only one of many possible tests of the hypothesis. Since the $\{N_n(\beta); \beta=1, \ldots, k(n)\}$ are asymptotically sufficient for our given class of alternatives, for any given sequence of tests there is a sequence of tests based on $\{N_n(\beta)\}$ which is asymptotically at least as good as the given sequence.

The asymptotic joint normality of $\{\overline{A}_n(j_1^{(i)}, \ldots, j_s^{(i)})\}$ can be proved directly, without using the theorem of Section 3. However, the theorem would be needed in analyzing the asymptotic properties of various other tests based on $\{N_n(\beta)\}$.

The whole development above assumes that the alternative distributions are approaching the distribution specified by the null hypothesis. Any reasonable sequence of tests should perform even better against alternatives which do not approach the distribution specified by the null hypothesis.

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A DYNAMIC NONLINEAR CONSTRAINED OPTIMAL REPLACEMENT MODEL

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ABSTRACT

A stochastically constrained optimal replacement model for capital equipment is constructed. Each piece of capital equipment, or machine, is characterized by its age and "utility" or "readiness" class. The readiness of a machine at any age is a stochastic function of its initial utility class and its age.

The total discounted replacement cost of several replacement streams, each commencing with an initial machine, is minimized with respect to the replacement age and initial utility class of each machine, subject to a readiness constraint stating the lower bound on the expected number of machines in each utility class at any time. A general solution procedure is outlined and a specific case is solved in detail.

INTRODUCTION

In [1] and [2] Kalman constructed a stochastically constrained optimal replacement model for capital equipment. A solution to this model was given under a number of restrictive assumptions, among them the abolition of the constraint. In this paper we will reformulate the constrained model in such a fashion that it can be solved using nonlinear programming techniques. By doing this, we will have achieved two improvements over the formulation in [1] and [2]:

(i) We will have succeeded in reformulating the dynamic, stochastic constraints in a fashion suitable for the application of a (numerical) solution procedure (previously we were only able to solve the unconstrained case), thus extending the model of [1] and [2].

(ii) The optimization is performed not only with respect to the lifetime of a piece of capital equipment, but also with respect to the "utility class" to which it belongs (previously we optimized with respect to the lifetime of a piece of capital equipment only).

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I. THE MODEL

We first define our notation, which closely agrees with that in [1]. We assume that there exi n incumbent machines and further that this number will remain constant. We use the followir notation:

u_j^i: the utilization rate of the *jth machine in the sequence of replacements starting with t ith initial incumbent machine* $i=1, \ldots, n$, $j=1, 2, \ldots$ (the italic phrase wi henceforth be denoted by $MRIM_{ij}$).

l_0^i: the age of the ith incumbent machine at time $t=0$.

l_j^i: the age of $MRIM_{ij}$ when it is replaced.

c_j^i: the utility class of $MRIM_{ij}$ when it is installed. ($c_j^i=1, \ldots, q$ in increasing order desirability).

$P_{kl}^{ij}(\tau)$: the probability, if $c_j^i=l$, that $MRIM_{ij}$ will be in utility class k when it is of age ($P_{kl}^{ij}(\tau)$ is assumed to be continuous in τ).

Before describing our model we will elaborate on one assumption we are to make. $MRIM$ will be replaced at age l_j^i, i.e. the old machine will be sold at its market value or disposed of some cost, and a new machine will be purchased in its place. We hence exclude any modernizatic of an incumbent machine. To incorporate the choice between replacement and modernization in our model would add several more decision variables and make the decision procedure even mo onerous, thus adding unnecessary complications to the analysis without, in our opinion, contributir much additional insight.

We will proceed to find an expression for the current account cost of an infinite sequence replacements of the ith incumbent machine, $MRIM_{i1}$. We first define the operating cost functic for $MRIM_{ij}$ to be

$$\varphi_j^i\left(u_j^i, \sum_{k=1}^{j-1} l_k^i - l_0^i + \tau, \tau, c_j^i\right) \tag{1}$$

where τ measures the age of $MRIM_{ij}$ and $\sum_{k=1}^{j-1} l_k^i - l_0^i + \tau$ is a measure of the absolute time t. W next define the investment cost of $MRIM_{ij}$ to be

$$W_j^i\left(c_j^i, \sum_{k=1}^{j-1} l_k^i - l_0^i\right) \tag{2}$$

where $\sum_{k=1}^{j-1} l_k^i - l_0^i$ is the absolute time of the purchase. Finally the salvage value of $MRIM_{ij}$ at the tin of its replacement is taken to be

$$S_j^i(c_j^i, \sum_{k=1}^{j} l_k^i - l_0^i, l_j^i). \tag{3}$$

The above choices of arguments, for functions representing operating costs, investment costs, an salvage values, will be suitable for many situations though not, of course, for all. A natural assum tion to make at this polnt is that

$$W_j^i(c_j^i, \sum_{k=1}^{j-1} l_k^i - l_0^i) > S_j^i(c_j^i, \sum_{k=1}^{j} l_k^i - l_0^i, 0),$$

i.e. the salvage value of brand new equipment is strictly less than the investment cost.

Under the assumption that replacements take place instantaneously we can now obtain an expression for the current accounts cost of the chain of machines starting with $MRIM_{i1}$:

$$w^i(l_1^i, \ldots, l_j^i, \ldots, c_2^i, \ldots, c_j^i, \ldots)\int_{l_0^i}^{l_{11}} \varphi_1^i(u_1^i, \tau, \tau, c_1^i)e^{-r\tau}d\tau$$
$$-S_1^i(c_1^i, l_1^i-l_0^i, l_1^i)\exp[-r(l_1^i-l_0^i)]$$
$$+\sum_{k=2}^{\infty}\exp\left[-r\left(\sum_{j<k} l_j^i-l_0^i\right)\right]\left\{\int_0^{l_k^i} d\tau\varphi_k^i(u_k^i \sum_{j=1}^{k-1} l_j^i-l_0^i+\tau, \tau, c_k^i)e^{-r\tau}\right.$$
$$\left.-\exp(-rl_k^i)S_k^i(c_k^i, \sum_{j\leq k} l_j^i-l_0^i, l_k^i)+W_k^i(c_k^i \sum_{j<k} l_j^i-l_0^i)\right\}, \quad i=1, \ldots, n,$$

where r is a projected constant discount rate. Of course it is possible to make r dependent on time by writing

$$r\left(\sum_{j=1}^{k} l_j^i-l_0^i+\tau\right),$$

but this would make our notation even more cumbersome. The interpretation of Equation (4) is straightforward and requires no further elaboration.

We now proceed to formulate a suitable constraint. First we will obtain an expression for the expected number of machines in utility class u at time t:

$$_u((l_r^s), (c_r^s); t)=\sum_{i=1}^{n}\sum_{m=1}^{\infty} P_{uc_m^i}^{im}(\tau)F\left(\sum_{k=1}^{m-1} l_k^i-l_0^i, \sum_{k=1}^{m} l_k^i-l_0^i, t\right), \quad u=1, \ldots, q, \quad \tau=t-\sum_{k=1}^{m-1} l_k^i-l_0^i.$$

where

$$F(a, b, t)=\begin{cases}1 \text{ if } a\leq t\leq b\\ 0 \text{ otherwise}\end{cases}.$$

The F function insures us that, if $MRIM_{im}$ is not in use at time t, i.e., if

$$t\notin\left[\sum_{k=1}^{m-1} l_k^i-l_0^i, \sum_{k=1}^{m} l_k^i-l_0^i\right],$$

the expression after the summation signs vanishes. However, if

$$t\in\left[\sum_{k=1}^{m-1} l_k^i-l_0^i, \sum_{k=1}^{m} l_k^i-l_0^i\right]$$

i.e., if $MRIM_{im}$ is in use at time t, this expression becomes

$$P_{uc_m^i}^{im}\left(t-\sum_{k=1}^{m-1} l_k^i-l_0^i\right)$$

which is the probability that $MRIM_{im}$ will be utility class u at time t, when its age is

$$\tau=t-\left(\sum_{k=1}^{m-1} l_k^i-l_0^i\right).$$

Since at time t, for a given i, only one $MRIM_{im}$ is in use, the summation over m will be a stream o zeroes except for one value equal to that in (7), when m is that of the $MRIM_{im}$ in use at time t. Hence the expression beginning with the second summation gives us the expected number of ma chines in utility class u at any time t in the replacement sequence commencing with $MRIM_{i1}$ fo any i. Finally, by summing over i we arrive at the total expected number of machines in utility class u at time t.

REMARK: $x_u(\cdot, \cdot, \cdot)$ as written is actually an upper semicontinuous correspondence in and (l_r^s), being multivalued when t and (l_r^s) are in the particular relation

$$t=\sum_{k=1}^{m} l_k{}^i - l_0{}^i$$

for some m. To make it a (single-valued) function we would have to change the range of definitio of $F(a, b, t)$ to the semiopen interval $a \leq t < b$. Such an $x_u(\cdot, \cdot, \cdot)$ would not be continuous though Since the set of multivaluedness of $x_u(\ldots)$, as defined is of measure zero, this multivaluedness i no way influences the interpretation of the model.

We will assume that the policy maker sets a policy of minimum utility by fixing a set o $\hat{x}_u(t)$, $u=1, \ldots, q$, the minimum expected number of machines in utility class u deemed acceptable where

$$\sum_{u=1}^{q} \hat{x}_u(t) \leq n.$$

DEFINE:

$$\zeta_u((l_r^s), (c_r^s); t) \equiv -\hat{x}_u(t) + x_u((l_r^s), (c_r^s); t) \qquad u=1, \ldots, q. \tag{8}$$

Then the constraint becomes

$$\zeta_u(\cdot\,; t) \geq 0 \text{ for all } t, \qquad u=1, \ldots, q, \tag{9}$$

or equivalently

$$\xi_u((l_r^s), (c_r^s)) \equiv \inf_{0 \leq t \leq T} \zeta_u(\cdot\,; t) \geq 0 \qquad u=1, \ldots, q, \tag{10}$$

where T is the time horizon of the policy maker, and possibly $T \to \infty$.

The problem of minimizing expenditure subject to a utility constraint can now be clearly defined. It is:

$$\min_{\substack{(l_r^s) \\ (c_r^s)}} \sum_{i=1}^{n} w^i(l_1{}^i, \ldots, l_j{}^i, \ldots, c_2{}^i, \ldots, c_j{}^i, \ldots) \equiv \min_{\substack{(l_r^s) \\ (c_r^s)}} w((l_r^s), (c_r^s)) \tag{11}$$

subject to

$$\xi_u((l_r^s), (c_r^s)) \geq 0$$

$$l_j{}^i \geq 0, \; c_j{}^i \epsilon [1, \ldots, q], \; \forall i \; \forall j$$

where

$$w(\ldots) \equiv \sum_{i=1}^{n} w^i(\ldots).$$

Apparently the problem as stated does not feasibly lend itself to an analytical solution for any large n or q except under assumptions reducing it to triviality. A numerical solution is also impossible since we have infinitely many decision variables $l_j{}^i$ and $c_j{}^i$. This difficulty can be overcome at the

ct of imposing appropriate additional hypotheses. The approach which most immediately suggests tself is to assume that for each i, $l_j{}^i$ and $c_j{}^i$ become independent of j for all sufficiently large j, i.e. o all j greater than some (known) j_i and j_i', i.e.,

1) $$l_j{}^i = l^i \quad j \geq j_i$$

1) $$c_j{}^i = c^i \quad j \geq j_i'.$$

F a discussion of this assumption, see [2] and [4].

We have now reduced the decision variables in (11) to a finite number

1) $$N = \sum_{i=1}^{n} j_i + \sum_{i=1}^{n} j_i'.$$

Te problem, as now stated, can in principle be solved to any desired degree of accuracy, using nlinear programming techniques, once the functional forms of the $\hat{x}_u$'s, the $\phi_j{}^i$'s, the $S_j{}^i$'s, the V_j's and the $P_{rk}{}^{ij}$'s as well as the values of the $u_j{}^i$'s, the $l_0{}^i$'s and r are known. Certainly, the ractical problems involved in determining and estimating all this information would typically be a from trivial, and in many cases the optimization calculations might prove to be unacceptably rublesome.

A SOLUTION PROCEDURE

We will obtain an analytical solution procedure under some further restrictions on the $w^i(\ .\ .\ .)$ uctions using a specific form for the probability function $P^{im}_{uc_m{}^i}(\ .\ .\ .)$. We make the following nitions.

DEFINITION 1: $k(i) \equiv \max\{j_i, j_i'\}$.

DEFINITION 2: $\tilde{w}^i(l_1{}^i, \ldots, l^i_{k(i)}; c_1{}^i, \ldots, c^i_{k(i)}) = w^i(l_1{}^i \ldots, l_j{}^i \ldots; c_1{}^i, \ldots, c_j{}^i, \ldots)$, ere $l_j{}^i = l^i_{k(i)}$ and $c_j{}^i = c^i_{k(i)}$ for all $j \geq k(i)$.

DEFINITION 3: $S((c_r{}^s)) \equiv \{(l_r{}^s) | \xi_u((l_r{}^s), (c_r{}^s)) \geq 0,\ l_j{}^i \geq 0,\ \forall u, i, j\}$.

We make the following assumptions:

ASSUMPTION 1: $\tilde{w}(l_1{}^i, \ldots, l^i_{k(i)}; c_1{}^i, \ldots, c^i_{k(i)})$ as a function of the $l_j{}^i$'s has continuous first tials.

ASSUMTION 2: For each i and given $c_j{}^i$'s, the function $\tilde{w}^i(l_1{}^i, \ldots, l^i_{k(i)}; c_1{}^i, \ldots, c^i_{k(i)})$ is siconvex on the set $\{(l_1{}^i, \ldots, l^i_{k(i)}) \epsilon R^{k(i)} | l_j{}^i \geq 0\}$.

LEMMA 1: The set $S((c_r{}^s))$ is closed.

PROOF: This follows directly from the upper semicontinuity of $x_u(\cdot, \cdot, \cdot)$.

REMARK: Assumption 1 follows from assuming smoothness of the $\varphi_j{}^i$'s, s, and $W_j{}^i$'s in the continuous-variable arguments $((l_r{}^s)$ and $\tau)$.

PROPOSITION 1: Under assumptions 1 and 2 and lemma 1, for a given $(c_r{}^s)$, the problem of ding an $(l_r{}^s)$ minimizing

$$\tilde{w}((l_r{}^s), (c_r{}^s)) \equiv \sum_{i=1}^{n} \tilde{w}^i(\ .\ .\ .)$$

subject to the constraint $\xi_u((l_r^s),(c_r^s)) \geq 0 \forall u \epsilon [1, \ldots, q]$, has the solution $(l_r^s)=(l_r^{s*})$, where (l_r^{s*}) is the solution to the unconstrained problem of minimizing $\tilde{w}((l_r^s), (c_r^s))$ with respect to (l_r^s), i.e. $(l_r^{s*}) \epsilon S((c_r^s))$; if not, the solution is on the boundary of $S((c_r^s))$.

PROOF: Trivial.

From Proposition 1 it follows that if an analytical expression for the boundary of $S((c_r^s))$ can be found, the problem stated therein is reduced to more manageable proportions. Let the solution be (l_r^s), $r=1, \ldots, k(s)$, $s=1, \ldots, n$; each l_j^i will be a function of (c_r^s). By substituting these functions back into $\tilde{w}((l_r^s); (c_r^s))$ we get a function $\hat{w}((c_r^s))$ which now can be minimized over the set of (c_r^s)'s for which $S((c_r^s)) \neq \phi$, giving the solution to the original problem (11).*

For this procedure to be feasible we thus need to be able to identify the set of (c_r^s)'s for which the set $S((c_r^s))$ is nonempty; and for each such (c_r^s) to find the unconstrained optimum of $\tilde{w}((l_r^s), (c_r^s))$ and to check whether it lies in $S((c_r^s))$. In case it does not, we need to be able to optimize over the boundary of $S((c_r^s))$ and finally to compare the minima for all feasible $(c_r^s))$ to obtain the minimum minimorum.

We will illustrate this procedure by a simple example. We assume the following:

$$
\begin{aligned}
l_j^i &= l^i && \forall j \forall i \\
c_j^i &= c^i && \forall j \forall i \\
u_j^i &= u^i && \forall j \forall i \\
l_0^i &= 0 && \forall i \\
P_{uc}{}^{ij} &= P_{uc}{}^i && \forall j \forall i \\
\varphi_j^i(\ldots) &= \alpha u^i + \gamma \tau + \delta c^i \\
W_j^i(\ldots) &= \epsilon c^i && (15) \\
S_j^i(\ldots) &= \eta c^i - \rho l^i \\
P_{uc^i}^i(\ldots) &= e^{-\pi_i \tau} \delta_{uc^i} + (1 - e^{-\pi_i \tau}) \delta_{u0} && (16)
\end{aligned}
$$

where

$$
\epsilon > \eta, \ \gamma > \rho r, \ \delta_{ij} = \begin{cases} 0 & i \neq j \\ 1 & i = j \end{cases}
$$

and $u=0$ means that the machine in question is useless. Now $w^i(l^i, c^i)$ becomes

$$
\begin{aligned}
w^i(l^i, c^i) = &\left[\frac{\alpha}{r} u^i + \frac{\delta}{r} c^i + \frac{\gamma}{r^2}\right] \\
&+ \frac{1}{e^{r l^i} - 1} \left[(\epsilon - \eta) c^i - \left(\frac{\gamma}{r} - \rho\right) l^i\right]
\end{aligned} \tag{17}
$$

It is easily verified that $w^i(\ldots)$ so defined satisfies assumption 2.† It can be shown‡ that

$$
\xi_u((l^s), (c^s)) = -\hat{x}_u + \sum_{i=1}^{n} e^{-\pi_i l^i} \delta_{uc^i}, \tag{18}
$$

where we have chosen the $\hat{x}_u$'s independent of time.

---

* S. Kaplan [3] has worked out a similar procedure for a one-dimensional case. We thank a referee for making us aware of this reference.

† This can be done by studying the first derivative of $w^i(\ldots)$ with respect to l^i which changes sign from negative to positive only once, a sufficient condition for quasiconvexity. It should be noted that although $w^i(\ldots)$ is quasiconvex, it is not convex.

‡ A proof, due to a referee, that this indeed occurs is included as Appendix A.

We next define a collection of sets, $I_1((c^s)), I_2((c^s)), \ldots, I_q((c^s))$, of natural numbers. We let $\epsilon_u((c^s))$ if and only if $c^i=u$. From this definition it is clear that $I_u((c^s)) \cap I_v((c^s))=\phi$, if $u \neq v$, and that

$$\{1, \ldots, n\}=\bigcup_{u=1}^{q} I_u((c^s)).$$

So (18) can now be written

$$\xi_u((l^s), (c^s))=-\hat{x}_u+\sum_{i \epsilon I_u((c^s))} e^{-\pi_i l^i}. \tag{3'}$$

We define a set $\Lambda((c^s))$ by

$$\Lambda((c^s))=\{(\lambda^s) \epsilon R^n | \forall u, \sum_{i \epsilon I_u((c^s))} e^{-\pi_i \lambda^i}-\hat{x}_u=0\}. \tag{4}$$

Using this, we find $S((c^s))$:

$$S((c^s))=\{(l^s) \epsilon R^n | (l^s) \geqq 0, \text{ and } (l^s) \leqq (\lambda^s) \text{ for some } (\lambda^s) \epsilon \Lambda((c^s))\}. \tag{5}$$

We note that in this example the problem is separable as both $S((c^s))$ and $w((l^s), (c^s))$ are separable. Thus, $S((c^i))$ can be written as the intersection of q sets

$$S((c^s))=\bigcap_{u=1}^{q} S((c^s), I_u((c^s))) \tag{6}$$

where

$$S((c^s), I_u((c^s)))=\{(l^s) \epsilon R^n | (l^s) \geqq 0, \text{ and } (l^s) \leqq (\lambda^s) \text{ for some } (\lambda^s) \epsilon \Lambda((c^s), I_u((c^s)))\} \tag{7}$$

with $\Lambda((c^s), I_u((c^s)))$ defined by

$$\Lambda((c^s), I_u((c^s)))=\{(\lambda^s) \epsilon R^n | (\lambda^s) \geqq 0, \text{ and } \sum_{i \epsilon I_u((c^s))} e^{-\pi_i \lambda^i}-\hat{x}_u=0\}. \tag{8}$$

We thus have q orthogonal hypersurfaces $\Lambda((c^s), I_u((c^s)))$, $u=1, \ldots, q$ each forming a boundary to the corresponding $S((c^s), I_u((c^s)))$. We define $\tilde{\Lambda}((c^s), I_u((c^s)))$ to be the projection of $\Lambda((c^s), I_u((c^s)))$ on the hyperplane $\{\lambda^i=0 \ \forall i \notin I_u((c^s))\}$. That is, on the coordinate hyperplane orthogonal to the hypersurface bounding $S((c^s), I_u((c^s)))$. We define $\tilde{S}((c^s), I_u((c^s)))$ in an analogous manner. We now see that

$$S((c^s))=\overset{q}{\underset{u=1}{X}} \tilde{S}((c^s), I_u((c^s))). \tag{9}$$

Since $w((l^s), (c^s))=\sum_{i=1} w^i(l^i, c^i)$, i.e. it separates additively in each l^i, the problem can now for any given (c^s) be broken down into q parts, one for each utility class:

$$\text{for } u=1, \ldots, q$$

$$\min_{(l^{s_u})} \sum_{i_u \epsilon I_u((c^s))} w^{i_u}(l^{i_u}, c^{i_u}) \tag{10}$$

subject to $(l^{s_u}) \epsilon \tilde{S}((c^s), I_u((c^s)))$ where the dimensionality of each (l^{s_u}) now of course is equal to the cardinality of $I_u((c^s))$.

Using (17) we first find each $l^{iu^*}(c^{iu})$ by equating the partial derivative of $w^i(\ .\ .\ .)$ with respec to l^i to zero and solving for l^i. This requires numerical solution of a transcendental equation. I $S((c^s)) \neq \emptyset$, i.e. if $\hat{x}_u \leq$ card $[I_u((c^s))]$ for all u, and if $(l^{su^*}(c^{su})) \epsilon \tilde{S}((c^s), I_u((c^s)))$, then $l^{iu^*}(c^{iu}) = \tilde{l}^{iu}((c^s))$. Assuming that $S((c^s)) \neq \emptyset$ and that $(l^{su^*}(c^{su})) \notin S((c^s))$, we find $(\tilde{l}^{su}((c^s)))$ by minimizing

$$\sum_{i_u \epsilon I_u((c^s))} w^{iu}(l^{iu}, c^{iu})$$

over the boundary of $\tilde{S}((c^s), I_u((c^s)))$, that is over $\tilde{\Lambda}((c^s), I_u((c^s)))$. Thus the problem becomes

$$\min_{(l^{su})} \sum_{i_u \epsilon I_u((c^s))} w^{iu}(l^{iu}, c^{iu}), \tag{26}$$

subject to

$$\sum_{i_u \epsilon I_u((c^s))} e^{-\pi i_u l^{iu}} - \hat{x}_u = 0$$

and

$$(l^{su}) \geqq 0.$$

The solution of (26) can occur (a) in the interior of the hypersurface defined by the constraints in (26), or (b) at a point where one or more of the l^{iu}'s is equal to zero, or (c), in the case the hyperplane is unbounded, i.e. the cardinality of $I_u((c^i))$ is greater than $(\hat{x}_u + 1)$, for a limiting process where one or more of the l^{iu}'s approaches infinity.

In case (a) the solution will occur at a point where the Lagrangian associated with (26),

$$L(\mu, (l^{su}); \ .\ .\ .) = \sum_{i_u \epsilon I_u((c^s))} [w^{iu}(l^{iu}, c^{iu}) - \mu e^{-\pi i_u l^{iu}}] - \mu \hat{x}_u \tag{27}$$

obtains a saddle point. The solution $(\tilde{l}^{su}((c^s)))$ is thus obtained as a solution to the system of equations obtained by equating the partial derivatives of L with respect to μ and all the l^{iu}'s to zero.

In case (b) we note that

$$w^{iu}(l^{iu}, c^{iu}) \to \infty \tag{28}$$

as

$$l^{iu} \to 0,$$

whereas $w^{iu}(l^{iu}, c^{iu})$ takes on finite values for all finite $l^{iu} \neq 0$. We can thus immediately rule out case (b).

In case (c) we note that the asymptotic behavior of the isocost surfaces generated by

$$\sum_{i_u \epsilon I_u((c^s))} w^{iu}(l^{iu}, c^{iu})$$

and the boundary of the constraint set is identical, i.e., the hypersurfaces are in both cases asymptotically perpendicular to the l^{iu} axis in the case of l^{iu} approaching infinity. A solution at infinity would thus be obtained by solving the Lagrangian problem outlined in case (a). We note that the form of the constraint that we have assumed in this specific example precludes a solution at infinity.

NOTE: A solution of l^{iu} at infinity means that the i_uth machine is dispensable and the total cost would be lowered by getting rid of machine i_u altogether.

Repeating the procedure outlined above for all values of u we obtain $(\tilde{l}^s((c^s)))$.

Again repeating the entire process for all values of (c^s) in $\{(c^s) \mid S((c^s)) \neq \emptyset\}$ and inserting the sting $(\tilde{l}^s((c^s)))$ into $w(\ldots)$, we obtain $\hat{w}((c^s))$ which is so minimized over the finite set $(c \mid S((c^s)) \neq \emptyset\}$.* We note that the cardinality of this set could be as high as nq, which for high ales of n and q might make this calculation unacceptably onerous.

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APPENDIX A

We here give a justification of Equation (18) for which we are indebted to an anonymous ee.

The expression in (18) is clearly a lower bound for all $\xi_u(\ldots; t)$. We will prove that this bound proached arbitrarily close. It is attained exactly when all replacements can take place simul- ously; i.e., when there exists a time t such that for all i, $t=n_i l^i$ for some integer n_i. This is pos- b if and only if the numbers l^i are commensurable.

The question is, when the l^i are incommensurable, can one find some t which comes arbitrarily o to simultaneously being integral multiples of all l^i? In other words, given small $\epsilon>0$, does there i a real number t and integers n_i such that

$$n_i l^i - \epsilon < t \leq n_i l^i$$

ll i?

In the branch of mathematics called Diophantine approximation, there is a theorem due to ir hlet which (in the form given on p. 98 of [A1]) asserts that, given real numbers $r_1, \ldots, r_n$, c $T>1$, there exists integers $p_1, \ldots, p_n$, and N with $1<N\leq T^n$, such that for all i

$$|r_i - p_i/N| \leq 1/TN.$$

pply this, let L be the product of the l^i and l the largest of them, and choose $T>\max\{1, 3l/\epsilon\}$. al $r=-L/l^i$. Then the above yields integers $p_1, \ldots, p_n$, and $N\geq 1$, with

$$|-L/l^i - p_i/N| < 1/TN < \epsilon/3Nl \leq \epsilon/3Nl^i.$$

Appendix B gives a numerical example worked out in detail illustrating this procedure.

With $n_i=p_i$, this yields

$$|n_i l^i - NL| < \epsilon/3.$$

Now divide the integers $i\epsilon\{1, 2, \ldots, n\}$ into two sets, writing $i\epsilon G$ ("good") if

$$n_i l^i - \epsilon/3 < NL \leq n_i l^i,$$

and $i\epsilon B$ ("bad") if

$$n_i l^i < NL < n_i l^i + \epsilon/3.$$

If all $i\epsilon G$, then we have the desired result, with $t=NL$. But if B is nonempty, then define $b\epsilon B$ b

$$n_b l^b = \min\{n_i l^i : i\epsilon B\},$$

and put $t=n_b l^b - \epsilon/3$. Then for $i\epsilon B$, one has

$$t < n_b l^b \leq n_i l^i$$

as well as

$$n_i l^i < NL < n_b l^b + \epsilon/3.$$

which implies

$$n_i l^i - \epsilon < n_b l^b - 2\epsilon/3 < t,$$

so that

(A1) $$n_i l^i - \epsilon < t < n_i l^i \qquad (i\epsilon B).$$

And for $i\epsilon G$ we have

$$t = n_b l^b - \epsilon/3 < NL - \epsilon/3 \leq n_i l^i - \epsilon/3 < n_i l^i$$

and

$$t = (n_b l^b + \epsilon/3) - 2\epsilon/3 > NL - 2\epsilon/3 > n_i l^i - \epsilon,$$

so that

(A2) $$n_i l^i - \epsilon < t < n_i l^i \ (i\epsilon G).$$

From (A1) and (A2) we have the desired result.

REFERENCE

[A1] Newman, M., *Integral Matrices* (Academic Press, 1972).

APPENDIX B

For illustrative purposes, we here give a numerical example of the solution procedure outlined in the text. We assume that there are four machines in use at all time (e.g. naval ships).

The operating cost in millions of dollars is given by

$$\phi^i = 1 + \tau + c^i,$$

where c^i is either 1 or 2 and τ is expressed in years. The purchasing cost in millions of dollars is taken to be

$$W^i = 10\ c^i,$$

the salvage value in millions of dollars is

$$S^i = c^i - l^i,$$

where l^i is expressed in years, and the probability function is taken to be

$$P^{i}_{uc\,t}=e^{\frac{-ir}{40}}\delta_{uc\,t}+\left(1-e^{\frac{-ir}{40}}\right)\delta_{u0};$$

ship one will have a half-life of 28 years; ship two, 14 years; ship three, 9.8 years; and ship , 7 years; the half lives reflecting the danger of the various assignments given the ships.

We take the interest rate r to be 10 percent. The readiness requirements are set at

$$\hat{\hat{x}}=\begin{pmatrix}0\\1.4\\1.1\end{pmatrix};$$

the expected number of ships in class 1 should not be less than 1.4 and in class 2 not less 1.1. Thus we have to plan to have two ships in each class; i.e., the set of (c^s)'s for which $S((c^s))$ nempty consists of six elements:

$$C_1=\begin{bmatrix}1\\1\\2\\2\end{bmatrix},\quad C_2=\begin{bmatrix}1\\2\\1\\2\end{bmatrix},\quad C_3=\begin{bmatrix}1\\2\\2\\1\end{bmatrix},\quad C_4=\begin{bmatrix}2\\1\\1\\2\end{bmatrix},\quad C_5=\begin{bmatrix}2\\1\\2\\1\end{bmatrix}\quad\text{and}\quad C_6=\begin{bmatrix}2\\2\\1\\1\end{bmatrix}$$

sets I_u then become:

$$I_0(C_k)=\phi\forall k$$

$I_1(C_1)=\{1, 2\}$	$I_2(C_1)=\{3, 4\}$
$I_1(C_2)=\{1, 3\}$	$I_2(C_2)=\{2, 4\}$
$I_1(C_3)=\{1, 4\}$	$I_2(C_3)=\{2, 3\}$
$I_1(C_4)=\{2, 3\}$	$I_2(C_4)=\{1, 4\}$
$I_1(C_5)=\{2, 4\}$	$I_2(C_5)=\{1, 3\}$
$I_1(C_6)=\{3, 4\}$	$I_2(C_6)=\{1, 2\}$

he sets $\tilde{S}$ become:

$$\tilde{S}(C_1, I_1)=\left\{\lambda^1, \lambda^2\geq 0 \middle| e^{\frac{-\lambda^1}{40}}+e^{\frac{-2\lambda^2}{40}}-1.4\geq 0\right\}$$
$$\tilde{S}(C_1, I_2)=\left\{\lambda^3, \lambda^4\geq 0 \middle| e^{\frac{-3\lambda^3}{40}}+e^{\frac{-4\lambda^4}{40}}-1.1\geq 0\right\}$$
$$\tilde{S}(C_2, I_1)=\left\{\lambda^1, \lambda^3\geq 0 \middle| e^{\frac{-\lambda^1}{40}}+e^{\frac{-3\lambda^3}{40}}-1.4\geq 0\right\}$$
$$\tilde{S}(C_2, I_2)=\left\{\lambda^2, \lambda^4\geq 0 \middle| e^{\frac{-1\lambda^2}{40}}+e^{\frac{-4\lambda^4}{40}}-1.1\geq 0\right\}$$
$$\tilde{S}(C_3, I_1)=\left\{\lambda^1, \lambda^4\geq 0 \middle| e^{\frac{-\lambda^1}{40}}+e^{\frac{-4\lambda^4}{40}}-1.4\geq 0\right\}$$
$$\tilde{S}(C_3, I_2)=\left\{\lambda^2, \lambda^3\geq 0 \middle| e^{\frac{-2\lambda^2}{40}}+e^{\frac{-3\lambda^3}{40}}-1.1\geq 0\right\}$$
$$\tilde{S}(C_4, I_1)=\left\{\lambda^2, \lambda^3\geq 0 \middle| e^{\frac{-2\lambda^2}{40}}+e^{\frac{-3\lambda^3}{40}}-1.4\geq 0\right\}$$
$$\tilde{S}(C_4, I_2)=\left\{\lambda^1, \lambda^4\geq 0 \middle| e^{\frac{-\lambda^1}{40}}+e^{\frac{-4\lambda^4}{40}}-1.1\geq 0\right\}$$
$$\tilde{S}(C_5, I_1)=\left\{\lambda^2, \lambda^4\geq 0 \middle| e^{\frac{-2\lambda^2}{40}}+e^{\frac{-4\lambda^4}{40}}-1.4\geq 0\right\}$$
$$\tilde{S}(C_5, I_2)=\left\{\lambda^1, \lambda^3\geq 0 \middle| e^{\frac{-\lambda^1}{40}}+e^{\frac{-3\lambda^3}{40}}-1.1\geq 0\right\}$$
$$\tilde{S}(C_6, I_1)=\left\{\lambda^3, \lambda^4\geq 0 \middle| e^{\frac{-3\lambda^3}{40}}+e^{\frac{-4\lambda^4}{40}}-1.1\geq 0\right\}$$
$$\tilde{S}(C_6, I_2)=\left\{\lambda^1, \lambda^2\geq 0 \middle| e^{\frac{-\lambda^1}{40}}+e^{\frac{-2\lambda^2}{40}}-1.1\geq 0\right\}$$

The function $w^i(l^i, c^i)$ now is

$$w^i(l^i, c^i)=110+10c^i)+\frac{9}{e^{\cdot 1 l^i}-1}(c^i-l^i).$$

Taking the derivative with respect to l^i and equating it to zero yields $l^{i_1*}(1)=4.831832$ and $l^{i_2*}(2)=7.067606$, i.e.,

$$(l^{i_1*}(c^{i_1}))=\begin{pmatrix}4.832\\4.832\end{pmatrix}$$

and

$$(l^{i_2*}(c^{i_2}))=\begin{pmatrix}7.068\\7.068\end{pmatrix}.$$

we find that

$$(l^{i_1*}(c^{i_1}))\epsilon\tilde{S}(C_k, I_1) \qquad k=1, 2, 3, 4, 5$$

$$(l^{i_1*}(c^{i_1}))\notin\tilde{S}(C_6, I_1)$$

$$(l^{i_2*}(c^{i_2}))\notin\tilde{S}(C_1, I_2)$$

$$(l^{i_2*}(c^{i_2}))\epsilon\tilde{S}(C_k, I_2) \qquad k=2, 3, 4, 5, 6.$$

Thus we have

$$(\tilde{l}^i(C_2))=(l^{i*}(C_2))=\begin{pmatrix}4.832\\7.068\\4.832\\7.068\end{pmatrix}$$

$$(\tilde{l}^i(C_3))=(l^{i*}(C_3))=\begin{pmatrix}4.832\\7.068\\7.068\\4.832\end{pmatrix}$$

$$(\tilde{l}^i(C_4))=(l^{i*}(C_4))=\begin{pmatrix}7.068\\4.832\\4.832\\7.068\end{pmatrix}$$

$$(\tilde{l}^i(C_5))=(l^{i*}(C_5))=\begin{pmatrix}7.068\\4.832\\7.068\\4.832\end{pmatrix}$$

$$(\tilde{l}^{i_1}(C_1))=(l^{i_1*}(C_1))=\begin{pmatrix}l^{1*}\\l^{2*}\end{pmatrix}=\begin{pmatrix}4.832\\4.832\end{pmatrix}$$

$$(\tilde{l}^{i_2}(C_6))=(l^{i_2*}(C_6))=\begin{pmatrix}l^{1*}\\l^{2*}\end{pmatrix}=\begin{pmatrix}7.068\\7.068\end{pmatrix}.$$

thus have to find

$$(\tilde{l}^{i_2}(C_1)) \text{ and } (\tilde{l}^{i_1}(C_6));$$

we have to minimize

$$w^4(l^3,2)+w^4(l^4,2)$$

ject to

$$e^{\frac{-3}{40}l^3}+e^{\frac{-4}{40}l^4}-1.4=0$$

l minimize

$$w^3(l^3,1)+w^4(l^4,1)$$

ject to

$$e^{\frac{-3}{40}l^3}+e^{\frac{-4}{40}l^4}-1.4=0.$$

find

$$(\tilde{l}^{i_2}(C_1))=\begin{pmatrix}\tilde{l}^3\\ \tilde{l}^4\end{pmatrix}=\begin{pmatrix}6.8860\\ 6.8644\end{pmatrix}.$$

$$(\tilde{l}^{i_1}(C_6))=\begin{pmatrix}\tilde{l}^3\\ \tilde{l}^4\end{pmatrix}=\begin{pmatrix}4.1516\\ 4.0413\end{pmatrix}.$$

ce we find $(\tilde{l}^i(C_1))$ and $(\tilde{l}^i(C_6))$ to be

$$(\tilde{l}^i(C_1))=\begin{pmatrix}4.832\\ 4.832\\ 6.886\\ 6.864\end{pmatrix} \qquad (\tilde{l}^i(C_6))=\begin{pmatrix}7.068\\ 7.068\\ 4.152\\ 4.041\end{pmatrix}.$$

stituting this back into $\sum_i w^i(l^i,c^i)$, we find

$$\hat{w}(C_1)=500+9\sum_i e^{\frac{c^i-\tilde{l}^i}{.1\,\tilde{l}^i}}=300.224$$

$$\hat{w}(C_k)=500+9\sum_i e^{\frac{c^i-\tilde{l}^i}{.1\,\tilde{l}^i}}=300.190$$

$$k=2,\ 3,\ 4,\ 5$$

$$\hat{w}(C_6)=500+9\sum_i e^{\frac{c^i-\tilde{l}^i}{.1\,\tilde{l}^i-1}}=301.134.$$

e an optimal policy is to use one ship of each class on the two most dangerous assignments and n ship of each class on the two least dangerous assignments. The ships in class one could then e placed after 4.8 years of service and the ships in class two after 7.1 years.

STATE-DEPENDENT GAP ACCEPTANCE*

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ABSTRACT

This paper considers a traffic problem in which vehicles queue up according to a Poisson process on a single-lane entrance ramp prior to merging into a major stream of vehicular traffic. In order to then prevent the ramp queue from becoming too large, a model is proposed which considers a lowering of the critical gap as the ramp queue size increases. With the critical gap assumed to be a nonincreasing function of the number of vehicles on the ramp at instances that correspond to departure times of lead vehicles from the ramp queue, the resultant model is an M/G/1 queue with state-dependent service times. Some general results are obtained for this model and a specific case discussed in moderate detail.

INTRODUCTION

The prototype model for the simple gap acceptance problem is the merging of a single lane of traffic into a major highway (see Drew [3]). Merging traffic has a yield sign and is supposed to give way to the traffic on the highway. This model is essentially the same as that for a normal intersection of a major and a minor road, and for a pedestrian trying to cross a one-way traffic stream. In all cases, gaps occur in the major stream flow, and for a merge (or cross) to occur a certain length of gap is necessary. Suppose the merging traffic can move only if a gap it faces has length T_0 or more, and thus the consecutive gaps occurring can be classified as either acceptable or not acceptable. When the main road traffic is moving freely, successive gaps can be considered to be independent of each other, and will thus be viewed as equivalent to the random sampling of a stochastic variable henceforth called T.

Based on this structure, it is easy to see that the probability of any operator having a wait of $A=m$ gaps before merging on the $(m+1)$st is the geometric distribution

$$p_m=(1-p)p^m \qquad (m=1, 2, \ldots),$$

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where, for $F(t)$ equal to the major stream-gap cumulative distribution function (CDF), $p=F(T_0)$. The total time of delay for this lead vehicle may thus be written as the random sum

$$D=\sum_{i=1}^{M} t_i$$

where the $\{t_i\}$ are the successive gaps faced. The *CDF* of D, say $B_D(t)$, is therefore constructed as a linear combination of convolutions of $F(t)$, namely,

$$B_D(t)=\sum_{m=0}^{\infty} p_m F^{*(m)}(t).$$

If now it is further assumed that the vehicles queue up in a Poisson fashion on a single-lane entrance "ramp", then the ramp may be described as an M/G/1 queue (ignoring possible waiting room limitations) with service times distributed as D, with *CDF* $B_D(t)$. Now it might become extremely important to prevent this queue (even in a stable environment) from getting too large because of possible downstream effects. This, in turn, might require the lowering of the critical gap as the queue increases. What is therefore being suggested is that the model permit the critical gap T_0 to be a nonincreasing function of the number on the ramp at the very instant a new vehicle becomes lead. Let us henceforth call this function $T_0^{(n)}$ and let D become D_n. It should be noted, furthermore, that it is not a very difficult matter to make the function $T_0^{(n)}$ completely arbitrary to allow for possible increases in the critical gap to relieve the pressure on the main traffic stream. All that we require mathematically for the model to remain valid is the preservation of the ramp queue stability.

This is not at all an unreasonable procedure since we can surely visualize a highway merging ramp where a signal is placed at the head of the ramp and is signalized in such a way to "permit" the lead vehicle to merge into gaps of decreasing width as the ramp becomes fuller. There is, of course, much precedence in the literature for the dynamic adjustment of ramp metering rates to balance the flow of traffic from the ramp onto the freeway (for example, see [9]). In fact, the idea of lowering critical gap settings as part of a ramp metering system has indeed been tried in the city of Houston. It works by the placement of a detector in the pavement which is actuated whenever the ramp queue exceeds some critical threshold (generally 14 or 15 vehicles) and then lowers the critical gap setting that is being used. The new gap stays in effect only as long as the queue remains above this critical level.

QUEUING MODEL

In view of the history of the use of Erlangian distributions as approximations for more general types of interarrival and service-time distributions in single-server systems (see Kendall [6]) and the papers by Harris in the *Naval Research Logistics Quarterly* [5], Rosenshine [8], and Kotiah, Thompson, and Waugh [7], the ramp queue shall be described as one with Poisson input, state-dependent Erlang service, and one server. Because of the reasonably favorable results the authors of [7] found in their evaluation of Erlangian approximations for $GI/G/1$, it is suggested here that a similar approach could be used for queues with Poisson input and more general but state-dependent service times and that these models should provide reasonable estimates.

The general state-dependent queuing system with Poisson input can be described in the following manner:

(1) Customers arrive as a simple Poisson process with parameter λ.
(2) The customers are serviced singly, first come, first served.
(3) The service time of each customer is conditioned on the number in the queue. Service times of customers beginning service with the same number n in the system are independent and identically distributed random variables, $\{D_n, n=1,2, \ldots\}$, with *CDF* $B_n(t) \equiv Pr\{D_n \leq t\}$.

Let X_k be the number of customers in the system immediately after the kth customer leaves the system. Then this (so-called) departure process is a Markov chain, independent of the form of the *CDF*'s $\{B_n(t)\}$ [4]. The transition matrix, $P=[p_{ij}]$, of this imbedded chain is given by

$$\begin{bmatrix} k_{01} & k_{11} & k_{21} & k_{31} & . & . & . \\ k_{01} & k_{11} & k_{21} & k_{31} & . & . & . \\ 0 & k_{02} & k_{12} & k_{22} & . & . & . \\ 0 & 0 & k_{03} & k_{13} & . & . & . \\ . & . & . & . & . & . & . \end{bmatrix},$$

where

$$k_{ij}=Pr\{i \text{ arrivals during a full-service period} | j \text{ were in the system when service began}\}$$

$$=(1/i!)\int_0^\infty e^{-\lambda t}(\lambda t)^i dB_j(t).$$

It is clearly seen for $i>0$ that $p_{ij}=k_{j-i+1,\,i}$, $(j \geq i-1)$, and that $p_{0j}=k_{j1}$.

The relationship now between the critical gaps $\{T_0^{(n)}\}$ for vehicles taking the lead with $n-1$ behind and the service procedure of the queuing problem can be found by the following argument. The service-time density is to be approximated by the Erlang, say

$$b_n(t)=\mu_n{}^{s_n} t^{s_n-1} e^{-\mu_n t}/(s_n-1)!,$$

in the Erlang parameters μ_n and s_n must therefore be estimated from the moments (whether empirical or actual) of the compound random variable D_n. An approximation would seem appropriate since the direct computation of the *CDF* of D_n leads to very messy expressions even in the simplest cases.

From the results contained in [4], it can be seen that the maximum-likelihood estimator for s_n is $\hat{s}_n/\bar{t}$, and that the *MLE* $\hat{s}_n$ is the best integer solution of the nonlinear equation

$$f(x)=0=k \log x+k\gamma+k/x-kx\sum_{i=1}^{\infty}[i(i+x)]^{-1}-k \log \bar{t}+\sum_{i=1}^{k} \log t_i,^*$$

* The infinite sum in $f(x)$ comes from the differentiation of a gamma function, and, if so desired, can be approximated for x not too small (say ≥ 3) by

$$kx\sum_{i=1}^{\infty}[i(i+x)]^{-1} \doteq k\left[\log\left(x-\frac{1}{2}\right)+\left(x-\frac{1}{2}\right)^2/24+\gamma+1/x\right].$$

where k is the number of observations, γ is Euler's constant, and t_i the observed ith service time. Since D_n is a random sum with M_n and all the component T_i mutually independent, its mean may be written as $E[D_n]=E[M_n]\ E[T]$. But M_n is geometric with mean $(1-p_n)/p_n$, and hence

$$E[D_n]=\frac{(1-p_n)}{p_n}\,E[T]$$

$$=\frac{1-F(T_0^{(n)})}{F(T_0^{(n)})}\,E[T].$$

Therefore if $\hat{s}_n/\hat{\mu}_n=\bar{t}$ is equated to $E[D_n]$, it is found that

$$\bar{t}=\frac{1-F(T_0^{(n)})}{F(T_0^{(n)})}\,E[T],$$

or that

$$F(T_0^{(n)})(\bar{t}+E[T])=E[T]$$

and

$$T_0^{(n)}=F^{-1}\left(\frac{E[T]}{\bar{t}+E[T]}\right).$$

To return to the $M/E_{s_n}/1$ queuing model, it can be seen from (1) that

$$k_{ij}=(1/i!)\int_0^\infty e^{-\lambda t}(\lambda t)^i\,\frac{\mu_j(\mu_j t)^{s_j-1}}{(s_j-1)!}\,e^{-\mu_j t}dt$$

$$=\frac{\lambda^i\mu_j{}^{s_j}}{i!(s_j-1)!}\int_0^\infty e^{-(\lambda+\mu_j)t}t^{i+s_j-1}dt$$

$$=\frac{\lambda^i\mu_j{}^{s_j}}{i!(s_j-1)!}\,\frac{(i+s_j-1)!}{(\lambda+\mu_j)^{i+s_j}}$$

$$=\binom{i+s_j-1}{i}(\lambda/\mu_j)^i/(\lambda/\mu_j+1)^{i+s_j}.$$

If the relative traffic intensity ρ_j is defined as $\rho_j=s_j\lambda/\mu_j$, then

$$k_{ij}=\binom{i+s_j-1}{i}(\rho_j/s_j)^i/(\rho_j/s_j+1)^{i+s_j}. \tag{2}$$

The chain is ergodic [2] when

$$\limsup_{j>0}\left\{\rho_j=\sum_{i=1}^{\infty} ik_{ij}=s_j\lambda/\mu_j\right\}<1.$$

When the steady-state distribution, $\{\pi_i\}$, of the system size exists, it is the solution of the system

$$\sum_{i=0}^{\infty} p_{ij}\pi_i=\pi_j \qquad (j=0,\ 1,\ 2,\ \ldots),$$

and

$$\sum_{i=0}^{\infty}\pi_i=1.$$

From the transition matrix,

$$\pi_j=\sum_{i=0}^{\infty} p_{ij}\pi_i$$

$$=k_{j1}(\pi_0+\pi_1)+\sum_{i=2}^{j+1} k_{j-i+1,\,i}\pi_i.$$

If (3) is multiplied by z^j and then summed on j, it is found that the probability generating ıction for the steady-state distribution $\{\pi_i\}$ is

$$\sum_{j=0}^{\infty}\pi_j z^j=(\pi_0+\pi_1)\sum_{j=0}^{\infty}k_{j1}z^j+\sum_{j=1}^{\infty}\sum_{i=2}^{j+1}k_{j-i+1,\,i}\pi_j z^j.$$

If the generating functions are defined as

$$\Pi(z)=\sum_{j=0}^{\infty}\pi_j z^j$$

$$K_i(z)=\sum_{j=0}^{\infty}k_{ji}z^j \qquad (i=0,\,1,\,2,\,\ldots),$$

ıı the aforementioned becomes

$$\Pi(z)=\pi_0K_1(z)+(1/z)[\pi_1 zK_1(z)+\pi_2 z^2K_2(z)+\pi_3 z^3K_3(z)+\ldots].$$

It should also be pointed out that all results for the imbedded chain can be directly related to ı general time process, $\{X(t),\ t\epsilon(0,\infty)/X(t)=$number of customers in the system at time $t\}$, re the stationary probabilities of the chain, $\{\pi_n\}$, are identical to those of $X(t)$ (see [4] for a rıf of this assertion).

Note that if $\Pi(z)$ is known in closed form, it is then a simple matter to obtain the expected ysem size L and waiting time W since

$$L=\Pi'(1)$$

$$=\lambda W.$$

lı tration

To illustrate, let the service distribution change only when the system size is greater than nre. That is,

$$b(t)=\begin{cases}\mu_1(\mu_1 t)^{s_1-1}e^{-\mu_1 t}/(s_1-1)! & (n=1,\,2,\,3)\\ \mu(\mu t)^{s-1}e^{-\mu t}/(s-1)! & (n\geq 4)\end{cases}.$$

Iece, from (2),

ı $Pr\{i$ arrivals during service time $\}\leq 3$ in systems when service began$\}$

$$=\binom{i+s_1-1}{i}(\rho_1/s_1)^i/(\rho_1/s_1+1)^{i+s_1}$$

k_i $Pr\ \{i$ arrivals during service time $\}>3$ in system$\}$

$$=\binom{i+s-1}{i}(\rho/s)^i/(\rho/s+1)^{i+s}.$$

Therefore the transition matrix is

$$P=\begin{bmatrix} k_{01} & k_{11} & k_{21} & k_{31} & . & . & . \\ k_{01} & k_{11} & k_{21} & k_{31} & . & . & . \\ 0 & k_{01} & k_{11} & k_{21} & . & . & . \\ 0 & 0 & k_{01} & k_{11} & . & . & . \\ 0 & 0 & 0 & k_0 & . & . & . \\ 0 & 0 & 0 & 0 & . & . & . \\ . & . & . & . & . & . & . \end{bmatrix}$$

and the generating functions are

$$K_1(z)=\sum_{j=0}^{\infty} k_{j1}z^j$$

and

$$K(z)=\sum_{j=0}^{\infty} k_j z^j.$$

Therefore, from Equation (4),

$$\Pi(z)=\pi_0K_1(z)+(1/z)\,[\pi_1zK_1(z)+\pi_2z^2K_1(z)+\pi_3z^3K_1(z)+\pi_4z^4K(z)+\pi_5z^5K(z)+\ldots]$$

$$=\pi_0K_1(z)+[K_1(z)/z]\,(\pi_1z+\pi_2z^2+\pi_3z^3)+[K(z)]\,[\Pi(z)-\pi_0-\pi_1z-\pi_2z^2-\pi_3z^3],$$

and thus

$$\Pi(z)=\frac{[zK_1(z)-K(z)]\pi_0+[K_1(z)-K(z)]\,[\pi_1z+\pi_2z^2+\pi_3z^3]}{z-K(z)}.$$

The probabilities π_0, π_1, π_2, and π_3 are obtained by first finding the limit of $\Pi(z)$ as z goes to one. Since $K_1(1)=1$, L'Hôpital's rule is used, and it is found that

$$\lim_{z\to 1}\Pi(z)=\frac{[K_1(1)+K'_1(1)-K'(1)]\,\pi_0+[K'_1(1)-K'(1)]\,[\pi_1+\pi_2+\pi_3]}{1-K'(1)}$$

$$+\frac{[K_1(1)-K(1)]\,[\pi_1+2\pi_2+3\pi_3]}{1-K'(1)}.$$

But $K_1'(1)=\rho_1$, $K'(1)=\rho$, and $K_1(1)=K(1)=\Pi(1)=1$, and hence

$$1=\frac{(1+\rho_1-\rho)\pi_0+(\rho_1-\rho)(\pi_1+\pi_2+\pi_3)}{1-\rho}.$$

Three more equations in π_0 and π_1 are obtained from (3), namely,

$$\text{(8)}\qquad \begin{cases} \pi_0=k_{01}(\pi_0+\pi_1) \\ \pi_1=k_{11}(\pi_0+\pi_1)+k_{01}\pi_2 \\ \pi_2=k_{21}(\pi_0+\pi_1)+k_{11}\pi_2+k_{01}\pi_3. \end{cases}$$

The four linear equations given by (7) and (8) can then be solved simultaneously for π_0, π_1, π_2, π_3, while the remaining $\{\pi_i\}$ would be obtained by iterating Equation (3). The stationary distribution would exist whenever $\rho<1$.

The computation of the $\{\pi_i\}$ would, of course, become rather cumbersome very quickly, so use will be made of the computer. Numerical values of the $\{\pi_i\}$ are displayed in Table 1 for some typical values of ρ and ρ_1, with s_1 and s fixed at the not unusual values of 3 and 5, respectively. Similar calculations can also be performed easily for arbitrary values of s_1 and s.

The two critical gaps, say $T_0^{(1)}$ and T_0, are found for each pair of rhos (assuming $\lambda=1$ with loss of generality and thus $L=W$) by

$$T_0^{(1)}=F^{-1}\left(\frac{E[T]}{s_1/\mu_1+E[T]}\right)$$

$$=F^{-1}\left(\frac{E[T]}{\rho_1+E[T]}\right)$$

$$T_0=F^{-1}\left[\frac{E[T]}{\rho+E[T]}\right]$$

TABLE 1.—*PI(I) VS. RHO, RHO1 Fixed*

RHO1=1.0					
RHO=	0.1	0.3	0.5	0.7	0.9
PI(0)	0.155138	0.147856	0.136337	0.115366	6.52113E-02
PI(1)	0.212597	0.202618	0.186832	0.158094	8.93637E-02
PI(2)	0.228132	0.217424	0.200485	0.169646	9.58938E-02
PI(3)	0.231757	0.220879	0.203671	0.172342	9.741 8E-02
PI(4)	0.108306	0.125113	0.138839	0.140454	9.43339E-02
PI(5)	4.18405E-02	5.44845E-02	7.44511E-02	9.71459E-02	8.54024E-02
PI(6)	1.48628E-02	2.07449E-02	3.47449E-02	6.11179E-02	7.41430E-02
PI(7)	5.00754E-02	7.30093E-03	1.48807E-02	3.65202E-02	6.31174E-02
PI(8)	1.62338E-03	2.44005E-03	6.02129E-03	2.12308E-02	5.33028E-02
PI(9)	5.10885E-04	7.85860E-04	2.34141E-03	1.21647E-02	4.48809E-02
PI(10)	1.57025E-04	2.46075E-04	8.84362E-04	6.91719E-03	3.77496E-02
PI(11)	4.73431E-05	7.53529E-05	3.26773E-04	3.91776E-03	3.17398E-02
PI(12)	1.40477E-05	2.26572E-05	1.18715E-04	2.21440E-03	2.66834E-02
PI(13)	4.11248E-06	6.70983E-06	4.25602E-05	1.25031E-03	2.24316E-02
PI(14)	1.19015E-06	1.96161E-06	1.50988E-05	7.05583E-04	1.88570E-02
PI(15)	3.41016E-07	5.67150E-07	5.31184E-06	3.98072E-04	1.58520E-02
PI(16)	9.68671E-08	1.62403E-07	1.85625E-06	2.24552E-04	1.33258E-02
PI(17)	2.73058E-08	4.61125E-08	6.45192E-07	1.26661E-04	1.12022E-02
PI(18)	7.64515E-09	1.29955E-08	2.23285E-07	7.14421E-05	9.41702E-03
PI(19)	2.12757E-09	3.63811E-09	7.70047E-08	4.02957E-05	7.91632E-03
PI(20)	5.88868E-10	1.01243E-09	2.64824E-08	2.27279E-05	6.65477E-03
$T_0(1)$	0.693147	0.693147	0.693147	0.693147	0.693147
T_0	9.53102E-02	0.262364	0.405465	0.530628	0.641854

TABLE 1.—Continued

RHO1=3.0					
RHO=	0.1	0.3	0.5	0.7	0.9
PI(0)	9.50972E-04	7.94371E-04	6.12745E-04	3.99574-04	1.45858E-04
PI(1)	6.65680E-03	5.56060E-03	4.28922E-03	2.79202E-03	1.02100E-03
PI(2)	4.18428E-02	3.49523E-02	2.69608E-02	1.25817E-02	6.41774E-03
PI(3)	0.260566	0.217658	0.167892	0.109483	3.99650E-02
PI(4)	0.223771	0.226562	0.210319	0.163967	7.11170E-02
PI(5)	0.162979	0.176151	0.184713	0.169487	8.87066E-02
PI(6)	0.111619	0.123296	0.139464	0.147759	9.42336E-02
PI(7)	7.33321E-02	8.17632E-02	9.71066E-02	0.116981	9.18487E-02
PI(8)	4.66651E-02	5.23200E-02	6.42670E-02	8.70222E-02	8.50696E-02
PI(9)	2.89514E-02	3.25927E-02	4.10507E-02	6.19479E-02	7.62923E-02
PI(10)	1.75948E-02	1.98734E-02	2.55304E-02	4.26630E-02	6.69902E-02
PI(11)	1.05117E-02	1.19062E-02	1.55466E-02	2.86286E-02	5.79965E-02
PI(12)	6.19047E-03	7.02850E-03	9.30533E-03	1.81190E-02	4.97323E-02
PI(13)	3.60124E-03	4.09727E-03	5.49003E-03	1.21483E-02	4.23699E-02
PI(14)	2.07298E-03	2.36281E-03	3.19959E-03	7.73118E-03	3.59388E-02
PI(15)	1.18235E-03	1.34982E-03	1.84510E-03	4.85901E-03	3.03931E-02
PI(16)	6.68939E-04	7.64788E-04	1.05424E-03	3.02108E-03	2.56516E-02
PI(17)	3.75772E-04	4.30166E-04	5.97491E-04	1.86076E-03	2.16206E-02
PI(18)	2.09747E-04	2.40385E-04	3.36197E-04	1.13665E-03	1.82067E-02
PI(19)	1.16409E-04	1.33552E-04	1.87959E-04	6.89271E-04	1.53227E-02
PI(20)	6.42743E-05	7.38101E-05	1.04477E-04	4.15268E-04	1.28904E-02
$T_0(1)$	1.38629	1.38629	1.38629	1.38629	1.38629
T_0	9.53102E-02	0.262364	0.405465	0.530628	0.641854

TABLE 1.—Continued

RH01=5.0					
RHO=	0.1	0.3	0.5	0.7	0.9
PI(0)	3.34382E-05	2.71142E-05	2:02280E-05	1.27013E-05	4.44028E-06
PI(1)	6.00650E-04	4.87051E-04	3.63355E-04	2.28153E-04	7.97605E-05
PI(2)	1.02012E-02	8.27187E-03	6.17107E-03	3.87485E-03	1.35462E-03
PI(3)	0.172831	0.140144	0.104552	6.56488E-02	2.29503E-02
PI(4)	0.170470	0.167544	0.150425	0.112921	4.69055E-02
PI(5)	0.146381	0.152106	0.151720	0.131443	6.46214E-02
PI(6)	0.120281	0.126896	0.133792	0.129603	7.50246E-02
PI(7)	9.58359E-02	0.101662	0.110468	0.116818	7.93519E-02
PI(8)	7.44872E-02	7.92601E-02	8.77006E-02	9.95165E-02	7.91914E-02
PI(9)	5.67220E-02	6.04962E-02	6.77669E-02	8.15073E-02	7.59649E-02
PI(10)	4.24614E-02	4.53737E-02	5.12995E-02	6.48227E-02	7.08182E-02
PI(11)	3.13295E-02	3.35335E-02	3.81976E-02	5.037732E-02	6.46212E-02
PI(12)	2.28314E-02	2.44727E-02	2.80533E-02	3.84236E-02	5.80025E-02
PI(13)	1.64613E-02	1.76668E-02	2.03630E-02	2.88518E-02	5.13937E-02
PI(14)	1.17581E-02	1.26332E-02	1.46319E-02	2.13784E-02	4.50739E-02
PI(15)	8.32994E-03	8.95882E-03	1.04210E-02	1.56602E-02	3.92087E-02
PI(16)	5.85849E-03	6.30639E-03	7.36410E-03	1.13570E-02	3.38826E-02
PI(17)	4.09366E-03	4.41015E-03	5.16787E-03	8.16380E-03	2.91247E-02
PI(18)	2.84386E-03	3.06595E-03	3.60414E-03	5.82238E-03	2.49274E-02
PI(19)	1.96528E-03	2.12015E-03	2.49955E-03	4.12328E-03	2.12606E-02
PI(20)	1.35168E-03	1.45907E-03	1.72473E-03	2.90149E-03	1.80821E-02
$T_0(1)$	1.79176	1.79176	1.79176	1.79176	1.79176
T_0	9.53102E-02	0.262364	0.405465	0.530628	0.641854

where $F(t)$ is again the gap *CDF*, with mean $E[T]$. For the purposes of this illustration, let it be assumed that $F(t)$ is the exponential e^{-t}, so that $F^{-1}(u) = -\ln u$. Hence $T_0^{(1)} = -\ln(\rho_1+1)^{-1} = \ln(\rho_1+1)$ and $T_0 = \ln(\rho+1)$.

Furthermore, the average queue size and waiting times may be found by differentiating the generating function $\Pi(z)$, and are thus found in terms of the generating functions $K(z)$ and $K_1(z)$. Or we may do the computation for L directly as $\Sigma n\pi_n$.

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THE EFFECT OF PRIOR ASSUMPTIONS ON THE EXPECTED COST OF M/M/1 QUEUEING SYSTEMS WITH UNKNOWN ARRIVAL RATE

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ABSTRACT

In this study we deal with the determination of optimal service rate in an M/M/1 queue. The arrival rate is unknown and assumed to be a random variable with a known distribution function. Holding and operating costs are considered and service rate is determined to minimize total expected discounted costs for infinite horizon. The effects of the arrival rate's distribution properties on the characteristics of the system are examined.

1 INTRODUCTION

Consider a service facility serving customers arriving at random. Holding and operating costs are taken into account. Operating costs are connected with the service facility and are assumed to be dependent on its capacity. Holding costs are caused by the time spent in the system by customers and may depend upon waiting times or the number of customers in the queue.

Common procedures for evaluating the optimal service capacity are based on the assumptions that arrival rate is known and is less than the service rate. Optimal capacity is usually determined to minimize the system's average costs per time unit. This capacity is sensitive to the arrival rate, particularly when they are close. In this case, a slight change in the flow of customers radically alters the queue size, and thus the holding costs.

Often, while planning queueing systems, the arrival rate is unknown and should be estimated or forecasted. The arrival rate may be estimated when the arrival process can be observed. Forecasting will be carried out when, while planning the system, the arrival process is yet unknown. For example, when a maintenance system is designed prior to operating the equipment, the failure rate

may be unknown. If estimation or forecasting is based on a large amount of information and the obtained service rate is not too close to the arrival rate, then replacement of this rate by its estimator or its forecast may be justified, otherwise it may lead to adverse results. If one can be sure that the arrival rate does not exceed a certain bound, then any capacity which is above this bound ensures steady-state conditions, hence finite average costs per time unit. Whenever such a bound does not exist, then for any service capacity saturation will occur with a certain positive probability. Here, it is impossible to obtain finite average costs.

This paper presents expected discounted accumulated costs for infinite horizon for evaluation of optimal service rate. We assume that customers arrive according to a Poisson input process with an unknown arrival rate and service times exponentially distributed (an M/M/1 queue). The arrival rate will be estimated or forecasted, and the result will be assumed to be a random variable with a known distribution (a Bayesian approach). Naturally, the more information available, the greater the concentration of the arrival rate's distribution around its expectation. The effect of this on the characteristics of the system will be examined. Bagchi and Cunningham [1] deal with a similar model, but in their work waiting space is finite.

2. THE BASIC MODEL

Consider a queueing system with a single server supplying services to customers arriving according to a Poisson input process with rate λ. Service times are independent random variables exponentially distributed with rate μ. The parameter μ will be referred to as the service capacity.

All future costs are discounted and e^{-pt}, $p>0$, is the present worth factor (PWF) by which any expenditure made at time t ($t>0$) is multiplied in order to evaluate its present worth (at $t=0$). We assume that the total expected discounted costs $C(\mu, \lambda)$ are the sum of the expected discounted operating costs $S(\mu)$ and the expected discounted holding costs $H(\mu, \lambda)$.

Operating costs are expenditures associated with the establishment and maintenance of the specified service capacity μ. These costs consist of setup costs $g(\mu)$—concerned with activation of the service capacity—and of service costs $s(\mu)$—expenditures of maintaining the service capacity for one time unit. We assume that $g(\mu)$, $\mu>0$, and $s(\mu)$ are monotone increasing differentiable and convex, furthermore, $s(0)=0$. The costs $g(\mu)$ may consist of certain fixed costs g_0 so that $g(\mu)$ may be discontinuous at $\mu=0$, ($g(0-)=0$, $g(0+)=g_0$). The operating costs are thus given by:

$$S(\mu)=g(\mu)+\int_0^\infty e^{-pt}s(\mu)dt=g(\mu)+\frac{s(\mu)}{p}. \tag{2.1}$$

Note that $S(0-)=0$, $S(0+)=g_0$ and $S(\mu)$ is assumed to be increasing differentiable and convex for all $\mu>0$.

Next, we assume linear holding costs of q per time unit for each queueing customer. This is a reasonable assumption and often made even if not valid. Denote by $P_y(t, \mu, \lambda)$ the conditional probability of having y, $y=0, 1, 2, \ldots$, customers in the system at time t, $t\geq 0$, given that the system is empty at time zero. The total expected discounted holding cost is then

$$H(\mu, \lambda)=q\int_0^\infty \sum_{y=0}^{\infty} yP_y(t, \mu, \lambda)e^{-pt}dt. \tag{2.2}$$

It is well known that

$$H(\mu, \lambda)=\frac{q}{p^2}(1-\xi)=\frac{q}{2p^2}\left[\lambda-\mu-p+\sqrt{(\lambda+\mu+p)^2-4\lambda\mu}\right], \tag{3}$$

where

$$\xi=\frac{1}{2\lambda}\left(\lambda+\mu+p-\sqrt{(\lambda+\mu+p)^2-4\lambda\mu}\right) \tag{4}$$

(see Saaty [3]).

The total conditional expected discounted costs (EDC) given that the queue is empty at time zero and the arrival rate equals λ is:

$$C(\mu,\lambda)=S(\mu)+H(\mu,\lambda)=\frac{s(\mu)}{p}+g(\mu)+\frac{q}{2p^2}\left[\lambda-\mu-p+\sqrt{(\lambda+\mu+p)^2-4\lambda\mu}\right]. \tag{5}$$

The optimal service rate, $\mu(\lambda)$, can be attained by simple methods of differential calculus. We give here a few remarks concerning the determination of this optimum. The function $H(\mu, \lambda)$ is strictly convex in μ. Furthermore, it was assumed that $S(\mu)$, $0<\mu<\infty$, is convex and monotone increasing. Accordingly, the equation $\frac{\partial}{\partial\mu}C(\mu, \lambda)=0$ has a unique solution μ' in $(0, \infty)$ for each λ such that $\frac{\partial}{\partial\mu}C(0+, \lambda)<0$. For all other λ we define $\mu'=0$. Thus the optimal service rate is $\mu(\lambda)=\mu'$ if $C(\mu'+, \lambda)\leq C(0-, \lambda)$ and $\mu(\lambda)=0$ otherwise.

It is interesting to notice that even in cases when $\mu(\lambda)>0$ it may be smaller than the arrival rate λ. This is due to the fact that $\frac{\partial}{\partial\mu}C(\mu, \lambda)$ is monotone increasing in μ. Hence, for any $0<\lambda<\infty$,

$$\frac{\partial}{\partial\mu}C(\lambda,\lambda)>0 \Rightarrow \mu(\lambda)<\lambda. \tag{6}$$

By differentiating $C(\mu, \lambda)$ we obtain, from (2.6), that $\mu(\lambda)\leq\lambda$ if

$$S'(\lambda)\geq\frac{q}{2p^2}\left[1-\frac{p}{\sqrt{p(4\lambda+p)}}\right]. \tag{7}$$

In other words, under certain cost conditions the optimal service rate may be smaller than the input rate (thus steady-state conditions are not met). In such cases, an unbounded accumulation of customers exists, because of high operating costs and a large discounting factor p.

3. THE CASE OF UNKNOWN INPUT RATE

Suppose the arrival rate is unknown. As stated in the introduction, we assume the arrival rate to be a random variable denoted by Λ, and let F be the (prior) distribution of Λ. The expected arrival rate is

$$\beta(F)=E[\Lambda]=\int_0^\infty \lambda dF(\lambda). \tag{8}$$

We consider only prior distributions, F, yielding a finite expectation β. As previously mentioned, replacement of λ by β in the formulae leading to derivation of the optimal service rate may yield erroneous results. We therefore replace the EDC, $C(\mu, \lambda)$, by the prior discounted cost, PDC, $K(\mu, F)$,

$$K(\mu, F)=\int_0^\infty C(\mu, \lambda)dF(\lambda)=S(\mu)+\int_0^\infty H(\mu, \lambda)dF(\lambda). \tag{3.2}$$

Following the convexity and differentiability of $C(\mu, \lambda)$, $K(\mu, F)$ is also convex and differentiable in μ except for a jump of magnitude g_0 at $\mu=0$. Indeed, since the function $H(\mu, \lambda)$ varies between zero and $q\lambda/p^2$, the derivative $\frac{d}{d\mu}K(\mu, F)$ is equal to

$$D(\mu, F)=\int_0^\infty \frac{\partial}{\partial\mu} C(\mu, \lambda)dF(\lambda). \tag{3.3}$$

Moreover, since $\frac{\partial}{\partial\mu} C(\mu,\lambda)$ is bounded by an integrable function of λ, then

$$\frac{d^2}{d\mu^2} K(\mu, F)=\int \frac{\partial^2}{\partial\mu^2} C(\mu, \lambda)dF(\lambda)\geq 0, \qquad \text{for any } \mu>0. \tag{3.4}$$

The Bayes optimal service rate $\mu^*(F)$ is, therefore, determined by the same procedure as $\mu(\lambda)$, by replacing $C(\mu, \lambda)$ by $K(\mu, F)$ and $\frac{\partial}{\partial\mu} C(\mu, \lambda)$ by $D(\mu, F)$. As in the case of a known arrival rate, this capacity may be smaller than the estimated arrival rate.

4. THE EFFECTS OF UNCERTAINTY

In this section we compare the expected cost and the optimal service rate of systems having the same cost parameters and the same expectation of the arrival rate. We compare a system with a known arrival rate to a system with an unknown arrival rate. We also compare two systems which differ in the prior distributions of the arrival rates.

The gamma and the normal distributions are frequently chosen as the prior distributions of the arrival rate. This is due to the conjugacy of the Poisson and the gamma distributions, and to the fact that the normal distribution is often applied in forecasting. Motivated by this observation, we assume that the distribution functions F_1 and F_2 of the second pair of systems that are compared have probability density functions $f_1(\lambda)$ and $f_2(\lambda)$ that satisfy the following two conditions:

$$\beta(F_1)=\beta(F_2)=\beta, \tag{4.1}$$

and two values, λ_1 and λ_2, $(0<\lambda_1<\lambda_2<\infty)$, exist such that

$$\begin{cases} f_1(\lambda)<f_1(\lambda), & \text{if } \lambda_1<\lambda<\lambda_2 \\ f_1(\lambda)>f_2(\lambda), & \text{if } 0<\lambda<\lambda_1 \text{ or } \lambda_2<\lambda. \end{cases} \tag{4.2}$$

The properties specified by (4.1) and (4.2) are satisfied by normal or gamma densities having a common mean β and with F_1 yielding a larger variance (or more uncertainty) than F_2.

We now prove the following lemma:

LEMMA:

A. Let f_1 and f_2 be two densities such that (4.1) and (4.2) are satisfied and let $u(\lambda)$, $0<\lambda<\infty$, be d integrable function (with respect to f_1 and f_2) satisfying

.3) $$u(\lambda)<l(\lambda),\ \lambda_1<\lambda<\lambda_2,$$
$$u(\lambda)\geq l(\lambda),\ \text{otherwise}$$

here $l(\lambda)$ represents the straight line through $(\lambda_1, u(\lambda_1))$ and $(\lambda_2, u(\lambda_2))$, then

.4) $$\int_0^\infty u(\lambda)f_1(\lambda)d\lambda>\int_0^\infty u(\lambda)f_2(\lambda)d\lambda.$$

B. Let F be a distribution function with expectation β. Let $u(\lambda)$ be an F-integrable function pported by a straight line $m(\lambda)$ such that $m(\beta)=u(\beta)$; i.e., $u(\lambda)\geq m(\lambda)$ for all λ and we assume at $u(\lambda)>m(\lambda)$ with positive probability (induced by F), then

.5) $$\int_0^\infty u(\lambda)dF(\lambda)>u(\beta).$$

PROOF:

A. $l(\lambda)$ is given by

.6) $$l(\lambda)=u(\lambda_2)+\frac{u(\lambda_2)-u(\lambda_1)}{\lambda_2-\lambda_1}(\lambda-\lambda_2),$$

mbining (4.1), (4.2), (4.3) and (4.6), we immediately obtain

.7) $$\int_0^\infty u(\lambda)[f_1(\lambda)-f_2(\lambda)]d\lambda>\int_0^\infty l(\lambda)[f_1(\lambda)-f_2(\lambda)]d\lambda=0,$$

hich proves part A.

B. The supporting line through β is given by

.8) $$m(\lambda)=u(\beta)+Q(\lambda-\beta),\qquad -\infty<Q<\infty.$$

is easily seen that

.9) $$\int_0^\infty u(\lambda)dF(\lambda)>\int_0^\infty m(\lambda)dF(\lambda)=u(\beta),$$

us proving part B.

COROLLARY: Conditions A and B hold when $u(\lambda)$ is strictly convex (in this case (4.9) is e Jensen's inequality). When $u(\lambda)$ is strictly concave, then inequalities (4.4) and (4.5) are reversed.

Since $C(\mu, \lambda)$, $\mu\geq 0$, is strictly convex in λ, we obtain from the corollary that

.10) $$C(\mu, \beta)<K(\mu, F) \text{ and } K(\mu, F_2)<K(\mu, F_1).$$

This means that for any assigned service capacity μ, the greater the uncertainty (variance), the higher the total expected costs. When the arrival rate is known, i.e., $\Lambda=\beta$, the expected discounted costs are minimal.

Next, we examine the effects of uncertainty on the optimal decision. These effects depend on the facts that $\frac{\partial}{\partial\mu}C(\mu,\lambda)$ is monotone decreasing in λ and

$$\frac{\partial^2}{\partial\lambda^2}\left(\frac{\partial}{\partial\mu}C(\mu,\lambda)\right)=2\frac{q}{p}[(\lambda+\mu+p)^2-4\lambda\mu]^{-5/2}[\lambda^2+\lambda(\mu+2p)-(2\mu-p)(\mu+p)]. \tag{4.11}$$

Hence, for any μ such that $0<\mu\leq p/2$, the function $\frac{\partial}{\partial\mu}C(\mu,\lambda)$ is strictly convex in λ for any $\lambda>0$, and for any $\mu>p/2$ a unique number λ^*, $(\lambda^*>0)$, exists such that $\frac{\partial}{\partial\mu}C(\mu,\lambda)$ is concave in λ for any $0<\lambda<\lambda^*$ and convex for any $\lambda>\lambda^*$.

Let $\mu^*(F_1)$ and $\mu^*(F_2)$ be the optimal service rates in two systems in which Λ is distributed according to F_1 and F_2, satisfying (4.1) and (4.2). The simplest and not very interesting case is when $0<\mu(\beta)$, $\mu^*(F_2)<p/2$. Here $\frac{\partial}{\partial\mu}C(\mu(\beta),\lambda)$ and $\frac{\partial}{\partial\mu}C(\mu^*(F_2),\lambda)$, $0<\lambda<\infty$, are strictly convex in λ and therefore, using the lemma and the corollary, we have

$$\frac{\partial}{\partial\mu}C(\mu(\beta),\beta)<D(\mu(\beta),F) \text{ and } D(\mu^*(F_2),F_2)<D(\mu^*(F_2),F_1) \tag{4.12}$$

for any distribution function F having expectation β. Following the optimization procedure we obtain $\mu(\beta)>\mu^*(F)$ and $\mu^*(F_2)>\mu^*(F_1)$. When $\mu(\beta)$ is not too large, then

$$\frac{\partial}{\partial\mu}C(\mu(\beta),0)\geq\frac{\partial}{\partial\mu}C(\mu(\beta),\beta)-\beta\frac{\partial}{\partial\lambda}\left(\frac{\partial}{\partial\mu}C(\mu(\beta),\beta)\right). \tag{4.13}$$

Hence,

$$\frac{\partial}{\partial\mu}C(\mu(\beta),\lambda)$$

is supported by its tangent through $\lambda=\beta$. Following part B of the lemma yields

$$D(\mu(\beta),F)>\frac{\partial}{\partial\mu}C(\mu(\beta),\beta) \tag{4.14}$$

for any distribution function F having expectation β. Thus, the optimal service rate for the case of an unknown arrival rate does not exceed $\mu(\beta)$. Furthermore, if $D(\mu^*(F_2),\lambda)$ satisfies the conditions of part A of the lemma, then we obtain $D(\mu^*(F_2),F_2)<D(\mu^*(F_2),F_1)$ and therefore $\mu^*(F_2)\geq\mu^*(F_1)$. When $\mu(\beta)$ appears sufficiently large with respect to β, then the probability that

$$\frac{\partial}{\partial\mu}C(\mu(\beta),\Lambda)$$

is convex is very small, and $D(\mu(\beta),F_j)$, $j=1,2$, behave as if

$$\frac{\partial}{\partial\mu}C(\mu(\beta),\lambda)$$

is strictly concave. Here, according to the corollary one obtains $\mu(\beta)<\mu^*(F_2)<\mu^*(F_1)$.

5. SOME NUMERICAL EXAMPLES

In order to illustrate the results obtained in Sections 2–4, we show two numerical examples. Let $q=1$ and

$$S(\mu)=b_1\mu^2+b_2\mu. \tag{1}$$

The arrival rate is assumed gamma distributed with $\beta=1$, namely,

$$\Lambda\sim\Gamma\left(\alpha,\frac{1}{\alpha}\right).$$

Uncertainty increases as α decreases. When $\alpha=\infty$, then $\Lambda=1$ with probability one. Let

$$F_\alpha=\Gamma\left(\alpha,\frac{1}{\alpha}\right).$$

The PDC $K(\mu, F_\alpha)$ [see (3.2)] was evaluated numerically according to the Gauss Legendre method.

EXAMPLE 1: Let $b_1=250$, $b_2=13500$, and $p=0.008$. Operating costs are high, hence $\mu(1)=0.78$ (system is saturated). Since $\frac{\partial}{\partial\mu}C(0.78, \lambda)$ is supported by its tangent at $\lambda=1$, the optimal service rate when the arrival rate is unknown is smaller than $\mu(1)$. Moreover, the optimal service rate decreases as uncertainty increases. (See Figure 1.)

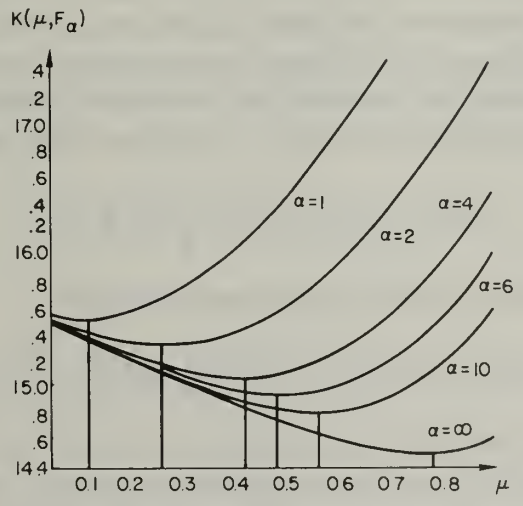

FIGURE 1. Expected costs of the system as a function of the service rate for Example 1 data.

EXAMPLE 2: Let $b_1=b_2=0.5$, $p=0.1$. Operating costs are low, hence $\mu(1)=2.6$. The probability that $\frac{\partial}{\partial\mu}C(2.6, \lambda)$ is convex is very small, hence the optimal service rate increases with uncertainty. (See Figure 2.)

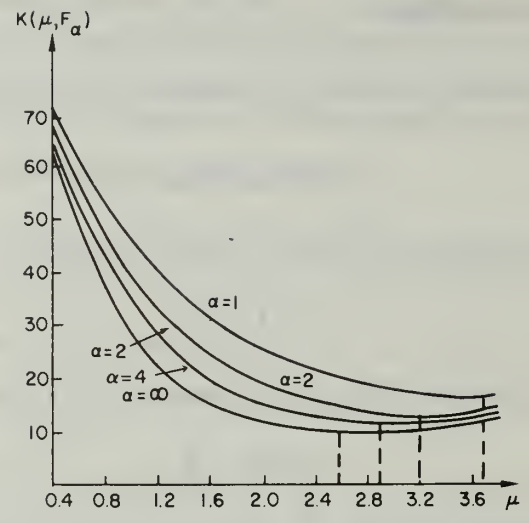

FIGURE 2. Expected costs of the system as a function of the service rate for Example 2 data.

6. DISCUSSION

The objective of the present paper is to show that cases exist where parameter of a queueing process should not be replaced by its estimator or its forecast.

In order to present this issue, the simplest possible model is preferred. An M/M/1 queueing system with a simple cost structure and an initial empty queue is discussed. The model can easily be generalized by assuming that: (a) The initial queue contains x customers ($x=0, 1, 2, \ldots$) (b) at commencement of service any customer pays the server a fixed amount of money—h. The total expected discounted cost, when assumption (a) is added to our basic model, is given by:

$$C(\mu, \lambda)=S(\mu)+\frac{q}{p}\left(x+\frac{\lambda-\mu}{p}\right)+\frac{q}{p}\frac{\xi^{x+1}}{1-\xi}. \tag{6.1}$$

When assumptions (a) and (b) are added, this expectation yields:

$$C(\mu, \lambda)=S(\mu)+\frac{q}{p}\left(x+\frac{1}{p}\right)-\left(\frac{q}{p}+h\right)\left(\frac{\mu}{p}-\frac{\xi^{x+1}}{1-\xi}\right). \tag{6.2}$$

The derivation of (6.1) and (6.2) is based on expressions given in [3]. All conclusions obtained in the discussed model remain valid for any of these modifications. However, the simple model is prefered since in our opinion these minor generalizations do not justify the algebraic complications involved in analyzing these models.

The result that the total expected discounted costs grow with uncertainty was expected, and intuitively acceptable. However, it was somewhat difficult to accept that under certain conditions optimization yields a reduction in service capacity as uncertainty grows. This difficulty is eliminated if one realizes that saturation is permitted, and when service is expensive, one is reluctant to establish a highly performing service facility.

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CONFIDENCE BOUNDS AND PROPAGATION OF UNCERTAINTIES IN SYSTEM AVAILABILITY AND RELIABILITY COMPUTATIONS*

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ABSTRACT

This paper develops bounds on the uncertainties in system availabilities or reliabilities which have been computed from structural (series, parallel, etc.) relations among uncertain subsystem availabilities or reliabilities. It is assumed that the highly available (reliable) subsystems have been tested or simulated to determine their unavailabilities (unreliabilities) to within some small percentages of uncertainty. It is shown that series, parallel and r out of n structures which are nominally highly available will have unavailability uncertainties whose percentages errors are of the same order as the subsystem uncertainties. Thus overall system analysis errors, even for large systems, are of the same order of magnitude as the uncertainties in the component probabilities. Both systematic (bias type) uncertainties and independent random uncertainties are considered.

INTRODUCTION

Calculations of estimates for large-system availabilities or reliabilities often use structural relationships (e.g., series, parallel, r out of n) among subsystem availabilities and reliabilities. , 7] Numerical values for the qualities of the subsystems are generally imprecise, known only with specified confidence intervals. For example, a finite duration test on a finite-sized sample of pe A subsystems can only give an *estimate* for the A-subsystem availability.

Those studies of component and system reliability and availability which have been based on al data have recognized the need for assessment of confidence bounds or other accuracy estimates. or example, Gnedenko et al. [1] describe confidence bounds on estimates of exponential failure tes. Rosenblatt [6] has described confidence calculations for system reliability based on tests or mulations of components in the system. That latter work makes extensive use of assumed forms r failure distributions. Mann et al. [4] summarize many recent results of that sort, and include an tensive bibliography. The work described here is motivated by the same kinds of problems as ose studied in [4] and [6]. However, we concentrate on simple inequality properties satisfied by ncertainties which arise when assessing highly reliable configurations, in contrast to the usual

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confidence interval calculations which use properties of particular failure distributions, asymptoti distributions, etc.

The purpose of this paper is to examine the way confidence intervals or uncertainties in sub system availabilities propagate through structural reliability calculations and ultimately induc uncertainties in the availability of the overall system. (*Reliability* uncertainty propagation prop erties are easily derived by obvious modifications of the results given here for availabilities.) Th general result is that, fortuitously, the percentage uncertainty in the overall system unavailability is the same order of magnitude as the largest percentage uncertainty in any subsystem unavail ability. These results are summarized in Tables 2.1, 3.1, 4.1.

Of the many possible types of subsystem availability uncertainties, we consider the two mos frequently encountered and simplest to analyze. These are bias (or systematic) errors; and zero- mean, random errors which are uncorrelated (from subsystem to subsystem).

Bias errors may arise from and relate to the following causes:

a) Subsystem availabilities are based on sparse and confusing test data. Assumptions are made concerning uniform interpretation of the test data, and the resulting working rule is seen to generally give overestimates or generally give underestimates.

b) A Monte Carlo calculation or an analytical solution is used to obtain the availability of a complex subsystem. A simplifying assumption may be used to more readily carry out the calculations, with corresponding uniformly optimistic or pessimistic results for the availability. An example of such bias errors would occur in a standby system with one on-line element and an imperfect switch permitting switch-over to the standby element. The assumption that the order of failure of the on-line element and the switch is im- material would result in an optimistic or pessimistic availability estimate, depending on the switch failure modes [5].

c) Although failure and repair rates may not be constant, they are often assumed to be con- stant, and the rates μ and λ are calculated as maximum likelihood estimates based on T hours of test data. Corresponding subsystem availability estimates are only reasonably valid at times comparable to the duration T of the test interval. For much longer or much shorter operating times a systematic error will appear in the calculated availability [2].

Zero-mean *random* uncertainties in subsystem availabilities seem appropriate in the following cases:

a) Attributes test data (i.e., r successes out of n items on test) on a subsystem's availability yield a mean and standard deviation. Due to the small size of the test sample, the coef- ficient of variation is large with a corresponding unpredictable uncertainty in the apparent availability.

b) Variables test data (i.e., r times-to-failure $t_1, t_2, \ldots t_r$ out of n items for T hours) is taken and the sample size is small.

c) The subsystem availability is estimated as an average over a small or modest number N of Monte Carlo simulations. This corresponds to either case a) or b) above.

Many other subsystem uncertainty models might be appropriate in particlar situations. For example, a combination of random and bias errors can occur as a result of finite sample size errors along with simplifying approximations, respectively. In particular, if estimates of μ and λ, and sampling distributions for these estimates, are derived from variables test data, then the limiting availability for subsystem k might be approximated by $A_k(\infty) \approx (1-\lambda/\mu)$, rather than the

xact value $A(\infty)=(1+\lambda/\mu)^{-1}$. This simplifies calculation of the sampling distribution for the vailability estimate, at the price of a bias error.

While only bias uncertainties and random uncertainties have been studied (separately), the ommon trend of the results given here suggests that more complicated types of subsystem uncer- ainties will propagate through structural calculations in a similar manner.

The derivations given below assume that all subsystems are of high quality, with very small navailabilities. Furthermore, the point of view is taken that subsystem testing or analytical evalu- tion should be sufficiently precise so that the percentage *unavailability* uncertainties are small. t seems more significant to normalize uncertainties to the nominal unavailabilities, rather than to ıe availabilities which will all be essentially unity for such purposes.

The following sections discuss series, parallel and r out of n structures, respectively. While ıe last structure includes the former two as special cases, the analytical methods used in that last ıse yield more restricted results. Tighter bounds are demonstrated for the series and parallel cases ıa the more specialized arguments which apply in each of those limiting cases.

. SERIES STRUCTURE

The main result of this section is that when the unavailability of a series connection of sub ystems is computed from uncertain estimates of subsystem unavailabilities, then the relative ncertainty of the system value is no greater than the largest relative uncertainty of any subsystem navailability. Corresponding results for availabilities and reliabilities are also developed.

We consider first bias, or nonrandom errors, using the following notation:

A_i=availability of ith subsystem,

$u_i=(1-A_i)$=ith system unavailability,

b_i=bias error in u_i (negative of bias in A_i),

$\bar{\bar{u}}_i$=nominal value of u_i.

ı this way

$$\bar{\bar{u}}_i=u_i+b_i, \tag{2.1}$$

nd the ith subsystem unavailability tolerance or uncertainty is

$$T_i=|b_i|/\bar{\bar{u}}_i. \tag{2.2}$$

For a series structure of N availability-independent subsystems, the system unavailability , is given by

$$\begin{aligned} u_s&=1-A_s \\ &=1-\prod_{i=1}^{N}(1-u_i) \end{aligned} \tag{2.3}$$

ith a similar relation among nominal values $\bar{\bar{u}}_i$ and $\bar{\bar{u}}_s$. Subsystems are availability-independent a repairman is always available when an equipment fails. For highly available systems this ondition will be approximately satisfied by a small number of repairmen.

A bound on the system tolerance

$$T_s=|u_s-\bar{\bar{u}}_s|/\bar{\bar{u}}_s \tag{2.4}$$

is obtained as follows. The main assumption here is that both the true and nominal subsystem unavailabilities are small

$$u_i \ll 1$$

(2.5a) $$\bar{\bar{u}}_i \ll 1$$

Moreover, since we are concentrating on total systems which are highly available, the *sums* of subsystem unavailabilities will also be small:

(2.5b) $$\sum_{i=1}^{N} u_i \ll 1.$$

The uncertainties need not be small (e.g., $T_i \approx 1$ is acceptable) so long as (2.5a, b) are satisfied. With this condition, approximations like

(2.6) $$\prod_{i=1}^{N} (1-u_i) \approx 1-\sum_{i=1}^{N} u_i$$

yield, from (2.3),

(2.7) $$u_s \approx \sum_{i=1}^{N} u_i$$

and similarly,

(2.8) $$\bar{\bar{u}}_s \approx \sum_{i=1}^{N} \bar{\bar{u}}_i.$$

The system tolerance as defined in (2.4) can then be written, using (2.1) and (2.7) for u_s and (2.8) for $\bar{\bar{u}}_s$, as

(2.9) $$T_s \approx |\sum \bar{\bar{u}}_i - \sum(\bar{\bar{u}}_i - b_i)| / \sum \bar{\bar{u}}_i = |\sum b_i| / \sum \bar{\bar{u}}_i$$

Using standard inequalities and substituting for $|b_i|$ from (2.2) we obtain

(2.10) $$|\sum b_i| \leq \sum |b_i| = \sum T_i \bar{\bar{u}}_i \leq T_{i_{\max}} \sum \bar{\bar{u}}_i$$

in which $T_{i_{\max}}$ is the largest subsystem tolerance. Thus, via substitution of (2.10) into (2.9) the final result on bias errors for a series structure becomes

(2.11) $$T_s \leq T_{i_{\max}},$$

i.e., the system unavailability uncertainty is less than the largest component unavailability uncertainty.

A similar result obtains when the uncertainties in subsystem unavailabilities for a series structure are modeled as *uncorrelated random errors*. In symbols

(2.12) $$u_i = \bar{u}_i + e_i$$

in which the errors e_i are independent and zero mean (where the notation $\bar{u}_i$ indicates the expected value of u_i), so that

(2.13) $$E[e_i e_j] = 0, \; i \neq j$$
$$= \sigma_i^2, \; i = j.$$

In this case the subsystem tolerance is defined as

(2.14) $$\tilde{T}_i = \frac{\sigma_i}{\bar{u}_i}.$$

If the subsystems are highly available with probability one, then the approximation in (2.7) ill still be applicable. The nominal system value $\bar{u}_s$ is then computed by taking the expectation f (2.3). The terms involving e_i and products of these errors vanish due to the zero-mean and ıdependence assumptions (c.f. (2.13)).

2.15) $$\bar{u}_s = 1 - \prod_1^N (1-\bar{u}_i)$$

$$\approx \sum_1^N \bar{u}_i.$$

he system unavailability variance is then approximated as

2.16) $$\sigma_s^2 \approx E[(\sum u_i - \sum \bar{u}_i)^2] = E[(\sum e_i)^2]$$

$$= \sum_1^N \sigma_i^2.$$

he last step makes use of the assumed independence properties of the random e_i.

Thus the system tolerance

2.17) $$\tilde{T}_s = \sigma_s / \bar{u}_s$$

approximated by

2.18) $$\tilde{T}_s \approx \frac{\sqrt{\sum \sigma_i^2}}{\sum \bar{u}_i}.$$

se of (2.14), bounding T_i by $T_{i_{\max}}$, and the triangle inequality for nonnegative $\bar{u}_i$ lead to

2.19) $$\sqrt{\sum \sigma_i^2} = \sqrt{\sum \tilde{T}_i^2 \bar{u}_i^2} \leq \tilde{T}_{i_{\max}} \sqrt{\sum \bar{u}_i^2} \leq \tilde{T}_{i_{\max}} \sum \bar{u}_i.$$

ombination of (2.18) and (2.19) shows that the random uncertainties in series-connected sub-stems produce a bounded system uncertainty

.20) $$\tilde{T}_s \leq \tilde{T}_{i_{\max}}$$

st as deterministic bias errors did. Actually, (2.20) is quite a pessimistic bound, because, e.g , all N subsystem unavailabilities are equal, then

$$\sqrt{\sum \bar{u}_i^2} = \sqrt{N} \bar{u}_i, \text{ and } T_i = T_{i_{\max}}.$$

Taking these substitutions in the first inequality in (2.19) and using (2.18) yields the tighter bound

2.21) $$\tilde{T}_s \leq \tilde{T}_i / \sqrt{N},$$

ıd the overall system unavailability uncertainty *decreases* as N increases. (This is understandable view of the concurrent increase in system unavailability.)

Availability uncertainties will now be computed in a similar fashion. While uncertainties in ıavailabilities are more meaningful measures of system performance, as demonstrated below, is mathematical analysis is presented because it will also be useful in deriving results in Section for the parallel structure unavailabilities.

In the case of deterministic *bias* errors, since we are discussing series systems with availabilit independence, system and subsystem availabilities satisfy the relations

$$A_s=\pi_1^N A_i \tag{2.22}$$

$$\overline{\overline{A}}_s=\pi_1^N \overline{\overline{A}}_i$$

in which the true and nominal values are related according to

$$A_i=\overline{\overline{A}}_i+b_i \tag{2.23}$$

in terms of the same bias error b_i used in (2.1).

The subsystem availability tolerances are defined as

$$T_i^1=|b_i|/A_i\ll 1 \tag{2.24}$$

and these are assumed to be very small. This assumption is consistent with the earlier one that bot the true and nominal systems must be highly available. This definition yields the system availabilit

$$A_s=\pi_i(\overline{\overline{A}}_i+b_i)=\pi_i\overline{\overline{A}}_i\left[\pi_i\left(1+\frac{b_i}{\overline{\overline{A}}_i}\right)\right] \tag{2.25}$$

which can be approximated using the inequality in (2.24) as

$$A_s\approx\pi_i\overline{\overline{A}}_i[1+\sum_j b_j/\overline{\overline{A}}_j]. \tag{2.26}$$

The system availability tolerance is

$$T_s^1=|A_s-\overline{\overline{A}}|/\overline{\overline{A}}_s. \tag{2.27}$$

The numerator of (2.27) is then found, using (2.26) and (2.22), and the fact that $\overline{\overline{A}}_i>0$ and (2.24)

$$|A_s-\overline{\overline{A}}_s|\approx|\pi_i\overline{\overline{A}}_i\sum b_j/\overline{\overline{A}}_j|=\pi_i\overline{\overline{A}}_i|\sum b_j/\overline{\overline{A}}_j|\leq\pi_i\overline{\overline{A}}_i\sum T_j^1. \tag{2.28}$$

Substitution of (2.22) and (2.28) into (2.27) shows the system availability tolerance to be bounde by the sum of the subsystem availability tolerances:

$$T_s^1\leq\sum_1^N T_j^1. \tag{2.29}$$

While this shows that system availability tolerances may grow as the number of series elements i increased, the nongrowing bound for the unavailability tolerance in (2.11) is more meaningful. I highly available systems with modest unavailability uncertainties, the corresponding availabilit uncertainties are orders of magnitude smaller. Therefore the sum in (2.29) will be small in absolut terms. Examination of the unavailability uncertainties is more informative in such cases.

The independent *random* error approach to propagation of availability uncertainties in a series tructure gives similar results. Using the same independent zero-mean random errors e_i as in 2.12) we have

2.30a) $$A_i=\overline{A}_i+e_i$$

nd the expected value of A_s is

2.30b) $$\overline{A}_s=E[\pi A_i]=\pi\overline{A}_i.$$

The random subsystem availability tolerance is defined as

2.31) $$\tilde{T}_i^1=\sigma_i/\overline{A}_i.$$

'he variance of the system uncertainty A_s defined in (2.22) is

2.32) $$\sigma_s^2=E[(\underset{i}{\pi}A_i-\underset{i}{\pi}\overline{A}_i)^2]=E[\underset{i}{\pi}A_i^2-2\underset{i}{\pi}A_i\underset{j}{\pi}\overline{A}_j+\underset{j}{\pi}\overline{A}_j^2].$$

ubstitution of (2.30a) and use of the independence of the errors e_i and A_i, along with substitution f (2.31) allows the simplification

2.33) $$\sigma_s^2=\pi_i(\sigma_i^2+\overline{A}_i^2)-\pi_j\overline{A}_j^2=\pi_i\overline{A}_i^2(\underset{j}{\pi}[1+(\tilde{T}_j^1)^2]-1).$$

'he assumption that all T_i^1 are small allows the inner product in (2.33) to be approximated by the rst two terms in the product expansion, with the result

2.34) $$\sigma_s\approx\pi\overline{A}_i\sqrt{\Sigma(\tilde{T}_j^1)^2}.$$

inally, the overall random error tolerance for availability of a series structure is

$$\tilde{T}_s^1=\sigma_s/\overline{A}_s.$$

Using (2.34) the approximate bound is

2.35) $$\tilde{T}_s^1\approx\sqrt{\Sigma(\tilde{T}_j^1)^2}\leq\Sigma\tilde{T}_j^1.$$

he upper bound here is effectively the same as in (2.29) for the corresponding bias error case. gain this bound is less meaningful than the corresponding unavailability bound.

The system tolerances derived in this section are summarized in Table 2.1.

TABLE 2.1. *Bounds on* Series *Structure System Tolerance T_s in Terms of Subsystem Tolerances T*

	Bias	Random
unavailability (small u_i, $\bar{u}_i$, $\bar{\bar{u}}_i$)	$T_s \leq (T_i)_{\max}$	$\tilde{T}_s \leq (T_i)_{\max} \dfrac{\sqrt{\Sigma \bar{u}_i^2}}{\Sigma \bar{u}_i} \leq (\tilde{T}_i)_{\max}$
		$\left(\leq \dfrac{(\tilde{T}_i)_{\max}}{\sqrt{N}} \text{ for equal } \bar{u}_i \right)$
availability (small T_i^1, $\tilde{T}_i^1$)	$T_s^1 \leq \sum_1^N T_i^1$	$\tilde{T}_s^1 \leq \sqrt{\sum_1^N (\tilde{T}_i^1)^2} \leq \sum_1^N \tilde{T}_i^1$
		$(\leq \sqrt{N} \tilde{T}_i^1 \text{ for equal } \tilde{T}_i^1)$

3. PARALLEL STRUCTURE

As a result of the dual roles of availability and unavailability with respect to series versus parallel structures, the four cases of bias or random uncertainties in availability or unavailability for parallel systems, can each be derived by simple reinterpretation of the results in the previous section. For example:

Parallel bias *un*availability: $u_s = \pi(\bar{\bar{u}}_i + b_i)$

Series bias availability: $A_s = \pi(\bar{\bar{A}}_i + b_i)$

Thus the result in (2.29) can be interpreted here for unavailabilities in a parallel structure of availability-independent elements as

$$T_s \leq \sum_1^N T_j. \tag{3.1}$$

This shows that system unavailability tolerance grows as the number of parallel elements is increased. Such growth will usually be kept within acceptable bounds since the T_j are usually quite small, and the number N of parallel redundant subsystems is for economic reasons usually a small integer.

Table 3.1 summarizes this result and the other three parallel-subsystem cases as found by similar reinterpretation of results from Section 2.

ABLE 3.1. *Bounds on* Parallel *Structure System Tolerance* T_s *in Terms of Subsystem Tolerances* T_i

	Bias	Random
unavailability (small T_i, $\tilde{T}_i$)	$T_s \leq \sum_1^N T_i$	$\tilde{T}_s \leq \sqrt{\sum \tilde{T}_i^2} \leq \sum_1^N \tilde{T}_i$
		($\leq \sqrt{N}\tilde{T}_i$ for equal $\tilde{T}_i$)
availability (small A_i, $\bar{A}_i$, $\bar{\bar{A}}_i$)	$T_s^1 \leq (T_i^1)_{\max}$	$\tilde{T}_s^1 \leq (\tilde{T}_i^1)_{\max} \frac{\sqrt{\sum \bar{A}_i^2}}{\sum \bar{A}_i} \leq (\tilde{T}_i^1)_{\max}$
		$\left(\leq \frac{(\tilde{T}_i)_{\max}}{\sqrt{N}}$ for equal $\bar{A}_i\right)$

. *r* OUT OF *n* STRUCTURE

This section considers a system made up of n subsystems in such a way that the system is navailable if less than r subsystems are available. Thus, a series structure corresponds to n out f n and a parallel structure corresponds to 1 out of n. We will show that if the subsystem unavailailities and their uncertainties are all small, then the bias case has the system fractional unavailbility bound

.1) $$T_s \leq (n-r+1)(T_i)_{\max},$$

ıd the random case has the bound

.2) $$\tilde{T}_s \leq \sqrt{n-r+1}\,(\tilde{T}_i)_{\max}.$$

hese general results are consistent with the results given for the series and parallel cases, but are ɔt as tight as the bounds given earlier for those cases.

We will use previously defined symbols along with the new term $p(j)=\{\alpha, \beta \ldots\}$ which notes the jth distinct collection of less than r subsystem indices $i=\alpha, \beta \ldots$. The system will unavailable when the only operating subsystems are those in a collection $p(j)$ of less than r of em. For example, for 3 out of 4 we may have $p(1)=\{1\}$, $p(2)=\{2\}$, . . . $p(5)=\{1, 2\}$, $p(6)=$, 3} . . . etc. The symbol $\bar{p}(j)$ will similarly denote those of the n indices which are *not* in $p(j)$. this way the *nominal* system unavailability may be written as the sum of the probabilities of ch disjoint situation in which the system is unavailable, namely

.3a) $$\bar{\bar{u}}_s = \sum_j \left(\prod_{i \in p(j)} (1-\bar{\bar{u}}_i) \prod_{i \in \bar{p}(j)} u_i \right).$$

The corresponding system unavailability may be expressed in terms of subsystem bias errors b_i

$$u_s=\sum_j\left[\prod_{i\in p(j)}(1-\bar{\bar{u}}_i+b_i)\prod_{i\in\bar{p}(j)}(\bar{\bar{u}}_i-b_i)\right]. \tag{4.3b}$$

Further notational simplicity is introduced by using π_j for $\prod_{i\in p(j)}$ and $\bar{\pi}_j$ for $\prod_{i\in\bar{p}(j)}$.

For example, in a 2 out of 3 case, one term in the sum of (4.3b) will be

$$\begin{aligned}\pi_1(i-\bar{\bar{u}}_i+b_i)\bar{\pi}_1(\bar{\bar{u}}_i-b_i)&=(1-\bar{\bar{u}}_1+b_1)\,(\bar{\bar{u}}_2-b_2)\,(\bar{\bar{u}}_3-b_3)\\ &=(1-\bar{\bar{u}}_1)\bar{\bar{u}}_2\bar{\bar{u}}_3+b_1\bar{\bar{u}}_2\bar{\bar{u}}_3-b_2(1-\bar{\bar{u}}_1)\bar{\bar{u}}_3-b_3(1-\bar{\bar{u}}_1)\bar{\bar{u}}_2\\ &\quad+\text{higher order terms in } b_i,\ b_j.\end{aligned} \tag{4.4}$$

The first order terms in b_i in (4.4) are scaled versions of the first (non-b_i) term, with scale factors of the form $\frac{b_i}{\bar{\bar{u}}_i}$ or $\frac{b_i}{(1-\bar{\bar{u}}_i)}$. The assumption of highly available subsystems means $\bar{\bar{u}}_i\ll 1$, so $\left|\frac{b_i}{\bar{\bar{u}}_i}\right|>\left|\frac{b_j}{1-\bar{\bar{u}}_j}\right|$. Therefore, $(r-1)$ of the linear-b_i terms are of negligible magnitude (e.g., $b_1\bar{\bar{u}}_2\bar{\bar{u}}_2$, in (4.4)). Moreover, the small uncertainty assumption $T_i=\left|\frac{b_i}{\bar{\bar{u}}_i}\right|\ll 1$ means that the higher-order b_i terms are relatively unimportant. Thus, for the example in (4.4) these approximations make only the first, third and fourth terms significant. Factoring out the first term and bounding T_i by $T_{i\,\max}$ yields

$$|\pi_1(1-\bar{\bar{u}}_i+b_i)\bar{\pi}_1(\bar{\bar{u}}_i-b_i)|\leq(1-\bar{\bar{u}}_i)\bar{\bar{u}}_2\bar{\bar{u}}_3[1+(n-r+1)\,(T_1)_{\max}]+\text{negligible terms}. \tag{4.5}$$

Equation (4.5) is a bound on a summand in (4.3b) having $(r-1)$ operative subsystems (the maximum number in an unavailable system). A summand with fewer operative subsystems will have even fewer than $(n-(r-1))$ significant linear b_i terms.

When all terms in the summation of (4.3b) are considered together, similar arguments yield

$$|u_s-\bar{\bar{u}}_s|\leq\bar{\bar{u}}_s[(n-r+1)\,(T_i)_{\max}+\text{higher order terms in the } T_i]. \tag{4.6}$$

Division by the nominal system uncertainty $\bar{\bar{u}}_s$ yields the approximate bound stated in (4.1).

Calculations for *random unavailability* uncertainty proceed in a similar manner, starting from

$$u_s=\sum_j\pi_j(1-\bar{u}+e_i)\bar{\pi}_j(\bar{u}_i-e_i) \tag{4.7}$$

in terms of the independent, zero mean subsystem uncertainties e_i. The e_i properties yield

$$\bar{u}_s=E[u_s]=\sum_j\pi_j(1-\bar{u}_i)\bar{\pi}_j\bar{u}_i. \tag{4.8}$$

The system tolerance

$$\tilde{T}_s=\frac{\sigma_{u_s}}{\bar{u}_s} \tag{4.9}$$

computed via

.10) $$\sigma_s^2 = E[u_s^2] - \bar{u}_s^2.$$

Separate consideration will be given to self- and cross-terms which arise when (4.7) is substituted into $E[u_s^2]$. A typical self-term D_j is of the form

(.11) $$D_j = E[\pi_j(1-\bar{u}+e_i)^2\bar{\pi}_j(\bar{u}_i-e_i)^2].$$

his can be evaluated using the independence properties of the e_i with the result

(12) $$D_j = \pi(1-\bar{u}_i)^2\left(1+\frac{\sigma_i^2}{(1-\bar{u}_i)^2}\right)\bar{\pi}\,\bar{u}_i^2\left(1+\frac{\sigma_i^2}{\bar{u}_i^2}\right) = \pi(1-\bar{u}_i)^2\bar{\pi}\,\bar{u}_i^2$$

$$+\text{terms linear in } \sigma_i^2$$

$$+\text{terms of higher order in } \sigma_i^2, \sigma_j^2, \ldots.$$

guments similar to those used in the bias uncertainty case show that the first-order terms in (.12) are dominated by the $(n-r+1)$ of them of the form

(.13) $$\frac{\sigma_k^2}{\bar{u}_k^2}\pi(1-\bar{u}_i)^2\bar{\pi}_i^2.$$

Thus, the quantity in (4.11) has the bound

(.14) $$|D| \le \pi_j(1-\bar{u}_i)^2\bar{\pi}_j\bar{u}_i^2[1+(n-r+1)\max\frac{\sigma_i^2}{\bar{u}_i} + \text{higher order terms in } \tilde{T}_i, \tilde{T}_j, \ldots].$$

A elf-term corresponding to less than $(r-1)$ operative subsystems will have even fewer significant correction terms.

When a typical cross-term B_j is considered,

(4.5) $$B_j = E\{[\pi_j(1-\bar{u}_i+e_i)\bar{\pi}_j(\bar{u}_i-e_i)][\pi_k(1-\bar{u}_i+e_i)\bar{\pi}_k(\bar{u}_i-e_i)]\}.$$

This can be written as

(4.6) $$B_j = \pi_j(1-\bar{u}_i)\bar{\pi}_j\bar{u}_i\pi_k(1-\bar{u}_i)\bar{\pi}_k\bar{u}_i$$

$$+ \text{ terms linear in } \sigma_i^2 + \text{higher order terms in } \sigma_i^2, \sigma_n^2, \ldots.$$

The terms linear in the variances σ_i^2 are like scaled versions of the first (nominal) terms in (4.16), with scale factors of the form

$$\frac{\sigma_i^2}{\bar{u}_i^2}, \frac{\sigma_i^2}{(1-\bar{u}_i)^2}, \text{ or } \frac{\sigma_i^2}{\bar{u}_i(1-\bar{u}_i)}.$$

The assumption of small subsystem unavailability uncertainty makes the latter two scale factors negligible compared to the first one. Thus the linear-variance terms are dominated by the less than $(n-r+1)$ of them with

$$\frac{\sigma_i^2}{\bar{u}_i^2}$$

scale factors, each of which is less than

$$\max_i \left(\frac{\sigma_i^2}{\bar{u}_i^2}\right) \pi_j(1-\bar{u}_i)\bar{\pi}\,\bar{u}_j\pi_k(1-\bar{u}_i)\bar{\pi}_k\bar{u}_i. \tag{4.17}$$

(These dominating terms correspond to e_i with identical subscripts in π_j and π_k. There are les than $(n-r+1)$ of these since the jth and kth sets of subsystems must be different.)

When each term in $E[u_s^2]$, formed with the aid of (4.7), is considered in the foregoing manne along with $\bar{u}_s$ defined in (4.8), the system variance defined in (4.10) has the bound

$$\sigma_i^2 \leq \max \left(\frac{\sigma_i^2}{\bar{u}_i^2}\right) (n-r+1)\bar{u}_s^2 + \text{higher order terms in } (\sigma_i^2/\bar{u}_i^2). \tag{4.18}$$

It follows that the system tolerance $\tilde{T}_s$ for random r out of n uncertainties is approximately bounde by

$$\tilde{T}_s \leq \sqrt{n-r+1}\ (\tilde{T}_i)_{\max} \tag{4.19}$$

Table 4.1 summarizes the bounds on system unavailability tolerances for bias and randoi uncertainties in subsystem unavailabilities, in each of the three structures.

TABLE 4.1. *Bounds on System* Unavailability *Tolerance in Terms of Subsystem Tolerances*

	Bias	Random
Series (n out of n) (Small $\bar{u}_i$, $\tilde{\bar{u}}_i$, u_i)	$T_s \leq (T_i)_{\max}$	$\tilde{T}_s \leq (\tilde{T}_i)_{\max} \dfrac{\sqrt{\sum \bar{u}_i^2}}{\sum \bar{u}_i} \leq (\tilde{T}_i)_{\max}$ $\left(\leq \dfrac{(\tilde{T}_i)_{\max}}{\sqrt{n}} \text{ for equal } \bar{u}_i\right)$
Parallel (1 out of n) (Small T_i and $\bar{u}_i$)	$T_s \leq \sum_1^n T_i$ (implies $T_s \leq n(T_i)_{\max}$)	$\tilde{T}_s \leq \sqrt{\sum \tilde{T}_i^2} \leq \sum_1^N \tilde{T}_i$ $(\leq \sqrt{n}\ \tilde{T}_i \text{ for equal } \tilde{T}_i)$ $(\leq \sqrt{n}\ (\tilde{T}_i)_{\max})$
r out of n (Small T_i, $\tilde{T}_i$ $\bar{u}$, $\tilde{\bar{u}}$)	$T_s \leq (n-r+1)(T_i)_{\max}$	$\tilde{T}_s \leq \sqrt{n-r+1}\ (\tilde{T}_i)_{\max}$

CONCLUSIONS

Uncertainties in system availabilities have been examined for several cases in which those vailabilities have been computed from estimates of subsystem availabilities. In all cases con- dered, percentage errors in the overall system unavailabilities are of the same order of magnitude s the percentage error in the least certain subsystem estimate. Quantitative bounds were found r each case.

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(R.Q) INVENTORY PROBLEM WITH UNKNOWN MEAN DEMAND AND LEARNING

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ABSTRACT

We address a single product, continuous review model with stationary Poisson demand. Such a model has been effectively studied when mean demand is known. However, we are concerned with managing new items for which only a Bayesian prior distribution on the mean is available. As demand occurs, the prior is updated and our control parameters are revised. These include the reorder point (R) and reorder quantity (Q).

Deemer, taking a clue from some earlier RAND work, suggested using a model appropriate for known mean, but using a Compound Poisson distribution for demand rather than Poisson to reflect uncertainty about the mean. Brown and Rogers also used this approach but within a periodic review context.

In this paper we show how to compute optimum reorder points for a special problem closely related to the problem of real interest. In terms of the real problem, subject to a qualification to be discussed, the reorder points found are upper bounds for the optimum. At the same time, the reorder points found can never exceed those found by the Compound Poisson (Deemer) approach. And they can be smaller than those found when there is no uncertainty about the mean.

As a check, the Compound Poisson and proposed approach are compared by simulation.

INTRODUCTION

We address a single product, continuous review model with stationary Poisson demand. ıch a model has been effectively studied when mean demand is known. However, we are concerned ith managing new items for which only a Bayesian prior distribution on the mean is available. s demand occurs, the prior is updated and our control parameters are revised. These include e reorder point (R) and reorder quantity (Q).

Deemer [4], taking a clue from some earlier RAND work [15], suggested using a model appro- iate for known mean, but using a Compound Poisson distribution for demand rather than Poisson reflect uncertainty about the mean. Brown and Rogers [1] also used this approach but within periodic review context.

Scarf [13], Iglehart [8] and Karlin [10] were concerned with characteristics of optimum solutions in the periodic review case. They did not develop computational algorithms nor did they consider setup cost. Murray and Silver [11], and Smith and Vemuganti [14] develop computational procedures for special cases of the periodic review model when there is not procurement lead time. The Murray-Silver paper treated the style goods inventory problem.

In this paper we show how to compute optimum reorder points for a special problem closely related to the problem of real interest. In terms of the real problem, subject to a qualification to be discussed, the reorder points found are upper bounds for the optimum. At the same time, the reorder points found can never exceed those found by the Compound Poisson (Deemer) approach. And they can be smaller than those found when there is no uncertainty about the mean!

As a check, the Compound Poisson and proposed approach are compared by simulation.

BASIC MODEL AND NOTATION

Demand is stationary Poisson with backlogging. There is a fixed lead time. There is a setup cost to buy, and backorder and holding costs are linearly time weighted functions of stock on hand or on backorder. The control parameters (R, Q) are updated every time there is a demand.

Solution of this model for known mean may be found in Hadley and Whitin for example [7]. It is the basic model used by the Department of Defense in the wholesale management of repair parts.

Crucial to this model is the ability to determine a backorder cost parameter, rather than to predetermine what the customer service level should be for each item. It is because of this that an item's reorder point can in some cases be lower when there is uncertainty about the mean than when there is not. For insight into how the Department of Defense determines an item's backorder cost parameter, see Presutti and Trepp [12], and for the actual details and modifications see [6].

NOTATION

d—mean demand rate
$f(d)$—prior on demand rate
C_h—holding cost per unit per year
C_b—backorder cost per unit per year
C_s—set up (administrative) cost of making a buy
L—lead time
$B(R, Q, d, L)$—steady state expected unit years of backorder per year for fixed R, Q
$O(R, Q, d, L)$—steady state expected on hand stock
$U(R, Q, d, L)$—steady state expected unavailability (proportion of time on hand is not positive)

REVIEW OF RESULTS FOR KNOWN MEAN

The objective is to minimize expected annual cost:

$$TC(R, Q)=C_s\frac{d}{Q}+C_bB(R, Q, d, L)+C_hO(R, Q, d, L). \tag{1}$$

owever,

$$O(R,\ Q,\ d,\ L)=R-(d)(L)+B(R,\ Q,\ d,\ L)+Q/2.$$

ence, substituting (2) into (1)

$$TC(R,\ Q)=C_s\frac{d}{Q}+(C_b+C_h)B(R,\ Q,\ d,\ L)+C_h(R-(d)(L)+Q/2).$$

For Poisson demand

$$B(R,\ Q,\ d,\ L)-B(R-1,\ Q,\ d,\ L)=-U(R,\ Q,\ d,\ L).$$

For given Q, the optimum R can be found from (3) by taking difference equations and using). Optimum R satisfies

$$(a)\qquad U(R^*,\ Q,\ d,\ L)\geq\frac{C_h}{C_b+C_h}$$

$$(b)\qquad U(R^*+1,\ Q,\ d,\ L)<\frac{C_h}{C_b+C_h}.$$

OMPOUND POISSON APPROACH TO UNKNOWN MEAN

Calculations of backorders reflects uncertainty about the mean:

$$B(R,\ Q,\ L)=\int f(d)B(R,\ Q,\ d,\ L).$$

Then equation (6) is substituted in (3) and consequently, R^* satisfies

$$(a)\qquad \int f(d)U(R^*,\ Q,\ d,\ L)\geq\frac{C_h}{C_b+C_h}$$

$$(b)\qquad \int f(d)U(R^*+1,\ Q,\ d,\ L)<\frac{C_h}{C_b+C_h}.$$

I have termed the approach Compound Poisson because the mathematics for solving (7) is te same as that for solving (5), except a Compound Poisson distribution, based on the prior (, is substituted for the simple Poisson. Computational procedures assuming a Gamma prior ıy be found in [4], and an approximation for higher mean demand items in [5].

The Compound Poisson approach is currently used by Army Materiel Command for a class o items. It would be optimum if R, Q were fixed rather than updated.

SPECIAL PROBLEM

Since we have a continuous review inventory system, the general problem of an optimum order point can be restated: What is the largest asset level at which it would be better to buy ow rather than to wait at least until the next demand? *This asset level is then the optimum* R. The

special problem differs from the general problem only in that it is assumed that if the next buy is delayed, quantity (Q) to be bought will *not* be revised based on when the next demand occurs.

Define $\Delta TC(R, Q)$ as the difference in total future cost if current assets are R, and a quantity Q is bought now rather than when the next demand occurs. Logically, then, the optimum R satisfies

$$\Delta TC(R^*, Q) \leq 0 \tag{8}$$

$$\Delta TC(R^*+1, Q) > 0.$$

For known mean demand, it is shown in [9]† that

$$\Delta TC(R, Q) = C_h \frac{Q}{d}[R-(R-1)] - (C_b+C_h)\frac{Q}{d}\left\{B(R-1, Q, d, L) - B(R, Q, d, L)\right\} \tag{9}$$

and that this is independent of future ordering policy.

Using (4),

$$\Delta TC(R, Q) = C_h \frac{Q}{d} - (C_b+C_h)\frac{Q}{d}U(R, Q, d, L) \tag{10}$$

As the problem is structured, the analogous equation holds for unknown mean:

$$\Delta TC(R, Q) = \int f(d) C_h Q/d - \int f(d)(C_b+C_h)\frac{Q}{d}U(R, Q, d, L). \tag{11}$$

Using (11) and (8), optimum R satisfies

$$(a) \quad \frac{\int \frac{f(d)}{d} U(R^*, Q, d, L)}{\int \frac{f(d)}{d}} \geq \frac{C_h}{C_b+C_h}$$

(12)

$$(b) \quad \frac{\int \frac{f(d)}{d} U(R^*+1, Q, d, L)}{\int \frac{f(d)}{d}} < \frac{C_h}{C_b+C_h}.$$

For known mean, (12) reduces to (5).

COMPUTATION

The Gamma prior has been cited as applicable to military repair parts [1, 4, 14].

Let

$$f(d; a, b) = \frac{b^a}{\Gamma(a)} d^{a-1} e^{-bd}, \qquad a > 0. \tag{13}$$

†[9] was a modest generalization of basic results used by Chern [2] in her paper on multi-item control. Karl Kruse played a major role in development of those particular results.

Then, the following is true:

(14a) Mean$=a/b$

(14b) Variance$=a/b^2$

(14c) $$\text{Coefficient of Variation}=\frac{1}{\sqrt{a}}$$

(14d) For $a>1$,

$$\left(\frac{1}{d}\right) f(d;\ a,\ b)=\frac{b}{a-1}\,\frac{b^{a-1}}{\Gamma(a-1)}\,d^{a-2}e^{-bd}$$

$$=\frac{b}{a-1}\,f(d;\ a-1,\ b),$$

nd

$$\int \frac{f(d;\ a,\ b)}{d}=\frac{b}{a-1}\int f(d;\ a-1,\ b)=\frac{b}{a-1}.$$

We will restrict attention to $a>1$, which requires only that the coefficient of variation be ess than 100 percent. Then, substituting (14d) into (12), R^* satisfies

15) $$\int f(d;\ a-1,\ b)U(R^*,\ Q,\ d,\ L)\geq \frac{C_h}{C_b+C_h}$$

$$\int f(d;\ a-1,\ b)U(R^*+1,\ Q,\ d,\ L)<\frac{C_h}{C_b+C_h}.$$

Comparison of (15) with (7) shows that for the Gamma prior with $a>1$, the only computa- ional difference between the proposed heuristic and the Compound Poisson approach is that the rior is first adjusted by decreasing the "a" parameter by 1. Inspection of (14a)–(14c) indicates ıe potential significance of this.

PPLICATION TO GENERAL PROBLEM

Since the special problem differs from the general problem only in that revision of Q is ex- uded when the next demand is received, the reorder point found is logically an upper bound ı the optimum. This rests on the premise that the ability to use additional information can only ıhance the value of delaying a buy. In practice, since we do not know how to compute an opti- um Q, the additional information could conceivably do more harm than good in some cases.

In addition to finding Q's, another unsolved problem is what to buy initially, a lead time fore the item goes into use. A theoretically correct procedure was developed, based on a sug- stion by Karl Kruse, which can choose the better of two alternatives: buy R only, or buy $R+Q$ ıppendix 2). Other alternatives could not be evaluated.

MULATION

The simulation was done as a check on the mathematics, and to see the magnitude of im- ovement. Regardless of the prior, R computed by the proposed heuristic will not exceed R com- ted by the Compound Poisson approach (Appendix 1). Since, subject to the qualification dis- ssed, the heuristic gives an upper bound on the true optimum, there is theoretical dominance.

Each replication of the simulation covered 5 years from initial use date of the item. Firs R^* and Q were computed based on the prior. Next mean demand for the replication was chos by sampling from the Gamma, and then demands were generated by sampling interarrival tim from the exponential distribution. The prior, R^*, and Q were updated each time there was a d mand, and an order was placed if necessary. If stock were below the reorder point—as could happ if R were revised upward—an amount equal to Q plus the deficit to the reorder point was bough Initially, a lead time before item use began, an amount of stock equal to R or $R+Q$ was bough depending on which led to lower expected cost (Appendix 2).

Costs—setup, backorder and holding—were computed first using the Compound Poisso approach to calculate R^* and then using the proposed procedure. Q was always calculated usin the classical Wilson formula with the mean of the updated prior as input.

Results are given in two tables. Table 1 describes the cases run in terms of a base case an eight others. Those input variables which were not given the base case values are indicated f each numbered case. In every case a holding cost rate of 25 percent of unit price, C_b of \$900, an a C_s of \$50 were used.

TABLE 1.

Case	Inputs				Initial Q	$\frac{C_h}{C_b+C_h}$
	AYD	Gamma "*a*"	*UP*	*PLT*		
Base	4	4	400	6 mos.	2	0. 100
1	2				1	0. 100
2	8				3	0. 100
3		2			3	0. 100
4		8			1	0. 100
5			200		3	0. 053
6			800		1	0. 182
7				3 mos.	2	0. 100
8				12 mos.	2	0. 100

TABLE 2.

Case	Initial R		Cost Saving	Percent	STD. DEV.
	Proposed	C.-P.			
Base	2	3	\$48. 00	2. 8	\$5. 3
1	1	1	\$1. 60	0. 1	5. 8
2	4	6	\$59. 00	2. 4	7. 3
3	1	3	\$119. 00	6. 5	9. 7
4	2	3	\$16. 00	1. 0	5. 2
5	3	3	\$13. 00	1. 2	4. 4
6	2	2	\$19. 00	0. 7	9. 9
7	1	1	\$10. 00	0. 8	4. 8
8	5	6	\$95. 00	3. 8	10. 7

Table 2 gives the results. Note that even if the two policies happened to give equal initial 's, they might compute different reorder points as the simulation progressed and demands were eceived. Hence costs could still differ.

The "percent" reported is the cost savings as a percent of annual costs. The standard deviaon is the simple standard deviation of cost differences.

Biggest saving is when uncertainty is greatest ("A" parameter=2). The savings are not large, ut are easily obtainable.

UMMARY

If a (Q,R) or order quantity, order point approach is to be the policy used, and if Q and R e fixed, then under Poisson demand, (and for a fixed lead time), expressions can be written for verage stock held and average back orders. The resulting cost expression can then be differentiated , in the discrete case, subjected to marginal analysis to find the optimal *fixed* Q^* and R^*. If the oisson mean is not known, but only a prior distribution exists, cost expressions have to be exected out, but the same analysis follows. This approach is *not* optimal for the case where future 's and R's may be changed after updating demand forecasts; however, it has seemed quite reasonble as a practical method and has so been used.

Our approach is now to look at the rather special case where Q *is not to be revised at least till ter the next demand occurs.* If this is the case, than at every stage one can try to find out whether buy (place an order) at the current stock level or to wait until next time. We then use the result at the cost difference between these two courses of action is independent of future ordering licy and are able to use the expression for this cost difference to decide on whether to order or ait.

The problem arises in that if we wait, then we are not entitled to revise Q on the next data int by assumption. Furthermore, we have to make the same assumption again to apply the proach. It would seem that we cannot by hypothesis revise Q till we actually place an order. om the practical point of view, however, this doesn't seem bad at all. Furthermore, there is thing in practice to prevent us from updating Q regularly but continuing as though we are not. e are inclined to think that this approach will compare pretty well with the optimal. In any ent we have proved it must always get reorder points as close or closer to the optimum than the order point calculated by the method heretofore recommended.

The real question then boils down to actual savings. The simulation results show small savings er the old method, but these are easily obtainable by adjusting one input parameter. Although e percentage savings are not great, the potential volume of application is large. Thus the new ethod is recommended.

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APPENDIX 1

RELATIVE SIZE OF THE R*'S

We will sketch the proof.

Let R_c be the solution of (7) and R_s the solution of (12). To prove $R_c \leq R_s$, it is enough to sho

$$\frac{\int \frac{1}{d} f(d) U(R, Q, d, L)}{\int \frac{1}{d} f(d)} \leq \int f(d) U(R, Q, d, L). \tag{A1.1}$$

To see this, note that by definition R_c satisfies (7). If (A1.1) holds, then since R_c satisfies (7b (12b) must also be satisfied if R^* is set equal to R_c there. If R_c also satisfies (12a), then by definiti $R_s = R_c$. Otherwise, to satisfy (12a), R^* must be set less than R_c.

The denominator on the L.H.S. of A1.1 equals $E_x(1/d)$ where E_x denotes expectation. I algebra, then, A1.1 is equivalent to

$$\int f(d) \left[\frac{1}{d} - E_x\left(\frac{1}{d}\right) \right] U(R, Q, d, L) \leq 0. \tag{A1.2}$$

Now if $U(R, Q, d, L)$ were constant for all d, the L.H.S. would equal 0 by definition of e pectation. But $U(R, Q, d, L)$ is monotonically increasing as a function of d. Hence all the negati terms in brackets are multiplied by higher values than the positive terms.

APPENDIX 2

Define $B(R, d, L)$ as expected value of time weighted backorders over a period L given tha initial on hand is R; mean demand rate is d; and no additional assets arrive during the perio

Then the net saving (possibly negative) of ordering Q with R a lead time before initial us date rather than placing a separate order for Q at that date is

$$C_s + C_b \int [B(R, d, L) - B(R+Q, d, L)] f(d) - C_h \{(Q)(L) - \int [B(R, d, L) - B(R+Q, d, L)] f(d) \tag{A2.1}$$

The equation is self-explanatory except for the holding cost term. This follows the logic use in [2].

Let

$O_1(t)$, $B_1(t)$—on hand, backorders respectively at time t if Q is bought with R

$O_2(t)$, $B_2(t)$—on hand, backorders if Q is not bought with R.

Then the basic idea behind the holding cost evaluation is that

$$Q = O_1(t) - B_1(t) - [O_2(t) - B_2(t)] \tag{A2.2}$$

or, by algebra,

$$O_1(t) - O_2(t) = Q - [B_2(t) - B_1(t)]. \tag{A2.3}$$

For computation of $B(R, d, L)$ see Chern [3], noting that for $L \neq 1$, μ must be replaced b $(\mu)(L)$.

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A SEMIECONOMIC METHOD FOR THE DETERMINATION OF OPTIMAL LOT SIZES

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ABSTRACT

The construction of lot sizes usually depends upon factors influencing homogeneity. When these factors are not a function of lot quantity, it is possible to determine an optimal lot size. The optimization process balances the cost of sampling against the expected cost of lot rejection for some specified procurement quantity. The rationale for balancing the two costs is contingent upon the fact that rejection criteria waivers frequently occur when the lot size is large. This concept implies that the lot size should be as small as possible, whereas the cost of sampling drives the lot size up. Hence, trade-offs may be made. The formulation is termed a semieconomic one because it combines a pure economic objective function with a pure statistical constraint. This constraint is necessary because the nature of the items under study dictates that the cost of accepting defective material cannot be explicitly stated. The paper presents the formulation, describes when it should be used, derives a good analytical approximation under certain assumptions and gives various ramifications when it is used.

. INTRODUCTION

The construction of lot sizes for lot-by-lot quality inspection in the Department of the Navy has traditionally followed the guidance set forth in reference [4]:

> "The formation of inspection lots is the procedure of assembling the units of product into identifiable lots, sublots, batches or in such other manner as may be prescribed. Each lot or batch shall, as far as is practical, consist of homogeneous units of product."

The usual process is to subjectively assess those cyclic parameters influencing the production process and then specify a temporal period which would include the bulk of those parameters in a homogeneous group. The quantity produced in this temporal period would be taken as the inspection lot size.

After the lots have been constructed, sampling inspection probabilistically determines acceptance or rejection of the lot. Various actions may be taken on rejected lots all of which have determinable, associated costs. Wetherill [11] has listed a number of actions which may be taken.

An additional action, which is of primary concern here, is to waive the rejection criteria and acce the lot, especially when the lot size is large. This facet of the managerial process introduces t topic of optimal lot size selection. Obviously, it is desirable to keep the sampling costs low, implyi large lot sizes, while keeping expected lot rejection costs low, implying small lot sizes.

There are factors other than the possibility of lot waivers that could plausibly force the l size down. The increase in storage costs as the lot size increases exemplifies this. However, none these factors are addressed.

The economic impact of quality control has been of interest for some time. Raiffa and Schlaif [9] have examined optimal-attribute, single-sampling plans when only sampling costs are considere Breakwell [1] presents an economical balance between test costs and inherent quality control cos for sequential tests and fixed size samples. Solomon and Zacks [10] give a comprehensive revie of the optimal design of sampling plans from finite populations. Hald [6] presents a model f finding optimal sample sizes and reject criterion using prior distributions and finite sampling. A of these papers, however, either assume that the lot size is fixed, very large, or that sampling wit replacement allows an infinite process, hence circumventing the lot size problem. This paper allow the lot size to vary in a fashion that will tend to balance inspection costs and expected costs of l rejection under a statistical constraint which will account for the intuitive costs of acceptin defective material.

II. ASSUMPTION FOR MODEL CONSTRUCTION

1. There may be various products involved when lot size optimization is desired. The mode presented in this paper is applicable for any product where the cost of accepting some proportio defective items cannot be explicitly determined (e.g., munitions, satellite components, jet part or pharmaceutical products).

2. The optimal lot size will be determined when single-sampling, attribute-inspection plan are applicable for determining product conformity during destructive tests.

3. The appropriate per unit cost coefficients are available.

4. The total quantity of items to be procured is specified and finite.

5. The production process is such that lot homogeneity is not greatly influenced by lot size

6. The storage costs for collecting the lot is negligible.

III. MODEL FORMULATION

The formulation includes the explicit cost of sampling and of lot rejection. The cost of accept ing some proportion defective material, as previously stated, cannot be explicitly assessed. Hence, the model will be what Case, Schmidt and Bennett [3] call a semieconomic approach (i.e., the formulation will be a mixture of pure statistical and pure economic criteria).

1. The Objective Function

The objective function is the pure economic portion of the formulation. Its construction must consider the trade-off of sampling costs and expected lot rejection costs for the entire pro-

urement quantity. It should not be a total cost minimization, such as the function set forth by rown and Rutemiller [2] which is:

1) $$f(N)=(nC_1)/N+(N-n)C_2(1-Pa)/N$$

here N=the lot size
n=the sample size
C_1=per unit sampling costs
C_2=per unit lot rejection costs
D=number of defective units in the lot
Pa=the probability of accepting D/N defective material.

It is obvious that the minimum of this function, since it is linearly decreasing as N increases, ill occur at K, the procurement quantity. This means N^* (the optimal lot size) will equal K nd only one sample would be taken. In light of the managerial tendency to waive rejected lots hen N^* is large, $N^*=K$ is an unsatisfactory solution.

Consequently, a more appropriate objective function would be one that minimizes the absolute ifference between sampling costs and expected lot rejection costs, or algebraically.

2) $$f(N)=|K\{(nC_1)-(N-n)C_2(1-Pa)\}/N|$$

Note that some managers would not want the break-even point between an actual cost and n expected cost to be the decision criterion because of extraneous knowledge (e.g., a particular anufacturer has always provided good quality in the past). They would be apt to weigh one of ne costs higher than the other. This concept may be easily incorporated into (2) by multiplying ne of the costs by a specific weight, W.

3) $$f(N)=|K\{(C_1n)-W(N-n)C_2(1-Pa)\}/N|$$

his function balances all of the sampling costs against W percent of the expected cost of lot ejection. The remainder of this paper will consider $W=1$.

. Constraint

The pure statistical portion of the formulation is given in one nontrival constraint. This onstraint merely restricts the set of N and n's to those values satisfying the cumulative hyper-eometric distribution. Before stating the constraint, note that it must intuitively incorporate the ost associated with accepting D/N proportion defective material. Duncan [5] and Yamane [13] iscuss the assessment of quality requirements when this cost is considered, but the process will e briefly reviewed.

Based on a myriad of reasons (experience being foremost), the assessor may feel that X percent efective material is the largest proportion defective material he can possibly accept without ncountering an exorbitant cost when a defective unit from an accepted lot is encountered. Even hen, he would only want to accept X percent a few times. This, basically, is the traditional con-umer parameter assessment of a typical sampling plan. Hence, the statistical constraint which uarantees the above quality requirements is;

$$F_h(C|n, D, N)=\sum_{i=0}^{C}\frac{\binom{D}{i}\binom{N-D}{n-i}}{\binom{N}{n}}=Pa \tag{4}$$

where $C\equiv$ the maximum number of acceptable sample defects (this is also part of the assessor input) The complete mathematical formulation, then, with all constraints is:

$$\min f(N)=|K\{nC_1-(N-n)C_2(1-Pa)\}/N| \tag{5}$$

subject to:

$$F_h(C|n, D, N)=\sum_{i=0}^{C}\frac{\binom{D}{i}\binom{N-D}{n-i}}{\binom{N}{n}}=Pa. \tag{6}$$

$$n\leq N\leq K; \tag{7}$$

n, N are integers.

IV. SOLUTION ALGORITHM

The constrained optimization problem just formulated has the distressing quality of not being completely defined until the solution has been found. This trait usually implies a dynamic program, but the formulation lacks an appropriate recursive relationship which Hillier and Lieberman [7] state is necessary for a dynamic program. Consequently, an iterative algorithm was developed to search the constraint set for those feasible N and n's that minimize $f(N)$. Obviously, the search technique employed will determine the efficiency of the algorithm. One of the faster (if not the fastest) search techniques available for one-dimensional, integer variable functions is the Fibonnaci search on a lattice, when the function is unimodal within the initial interval of uncertainty [8]. This method is described in [12] and will not be discussed in detail here.

The steps of the algorithm are, briefly:

1. Check K to determine if it is a Fibonnaci number minus one. If not, the next higher Fibonnaci number, K', must be found.

2. The Fibonnaci search is now conducted on the interval $(0, K')$ for N.

3. At each test point N of the search, the objective function is evaluated where n satisfies the constraints.

4. The procedure continues until N^* is found.

V. AN APPROXIMATION

A good approximation can be made to the optimal lot size under certain conditions. If C_1 is considerably larger than C_2, then C will probably equal 0, for this accept number gives the smallest sample size for fixed quality requirements. Also, if K is relatively large, the above cost relationship will tend to force n to be small in relation to N. Hence, the binomial distribution may be used. Recall that the sampling costs are to be balanced with the expected cost of lot rejection, or:

8) $$(K/N)nC_1=(K/N)(N-n)C_2(1-Pa)$$

9) $$n=[NC_2(1-Pa)]/[C_1+C_2(1-Pa)].$$

The binomial equation may be written (under the above assumptions) as:

10) $$Pa=p^0(1-p)^n$$

11) $$n=\ln Pa/\ln(1-p).$$

Equations (9) and (11) yield:

12) $$N=[\ln Pa(C_1+C_2-C_2Pa)]/[\ln(1-p)(1-Pa)C_2]$$

VI. AN EXAMPLE

Suppose the Department of the Navy decides to purchase 150,000 mechanical time fuzes for use on a particular projectile. It has the authority to specify the size of each lot submitted for quality inspection. The test apparatus has been used for some time (consequently, there is no fixed test cost). The evaluated per unit test cost is \$700 per fuze. The cost of lot rejection (when rejection precipitates replacement of the fuze) is \$17 per fuze, including labor. The intuitive cost of accepting defective material is sufficiently high that upon lot acceptance the statistical requirements of being 95 percent confident that the reliability is at least 95 percent have been specified with $C=0$. From these data:

$C=0$
$C_1=\$700$
$C_2=\$17$
$K=150{,}000$
$Pa=.05$
$D=.05N$ (raised to the next higher integer if a decimal occurs).

The algorithm gives a solution:

$$N^*=2526.$$

The assumptions for use of the approximation are satisfied by this example, and its solution is:

$$N^*=2590.$$

VII. SUMMARY AND REMARKS

Under the assumptions delineated, it is possible to determine a lot size that balances the sampling costs against the expected costs of lot rejection, optimally. It should be noted that K/N, intentionally, has not been restricted to an integer. The reason is that K very likely has few common divisors (or even may be prime), which greatly reduces the feasible N. The consequences of not restricting K/N to an integer is that there undoubtedly will be a quantity of units, R, such that $R<N$, remaining after K has been divided into groups of N. The algorithm will compute the costs of sampling and expected cost of lot rejection in a fractional manner. Caution should be exerted because the quality requirements may be violated by $(R/N)n$ samples being taken

from a lot of R when R is small relative to N. If this happens, a sub-optimal approach should be taken. The first possible method would be to increase the last N by R and retain the same sampling plan. Otherwise, it would be necessary to construct an n for R that guarantees the specified quality requirements.

A further constraint may be needed in the formulation when $C_1 << C_2$. The reason is that this condition forces N^* to be very small. When this happens, a large portion of N is sampled to meet the requirements. Since this sampling is often destructive, few items are left for use. In cases like these, the constraint

$$\sum_{i=1}^{K/N} (N_i - n_i) = \text{usage requirements}$$

should be included in the formulation to guarantee that the unit requirements are met.

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STOCHASTIC DUELS WITH BURSTS*

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ABSTRACT

In this analysis we extend the theory of stochastic duels to include the situation where one of the contestants fires a single infinitely long burst at a fixed rate of fire and secondly where he fires fixed-length, fixed-rate of fire bursts randomly spaced. In both cases the opponent fires single rounds randomly spaced. Special cases and parameter effects are considered.

NTRODUCTION

The fundamental stochastic duel, as originally defined, matches two contestants A and B. ach starts with unloaded weapons and each fires periodically. Both have unlimited ammunition pplies, unlimited time, and fixed kill probabilities, p_A and p_B respectively. The case where each ntestant has a random continuous time between rounds was considered first [3] and the case here each has a fixed time between rounds was done later [1]. The first duel is a model where ch contestant has a nonautomatic weapon, and the second is one where each fires an automatic eapon continuously (a single long burst). Many modifications and extensions have been added ce then [2].

In this paper, we extend the solution of the fundamental duel by first considering the situation here one duellist has a fixed firing time (automatic weapon firing one long burst continuously) d the other has a random firing time (nonautomatic weapon). Then, the more realistic situation here automatic weapons are fired in fixed length bursts with the bursts being randomly spaced considered.

IXED CONTINUOUS AND FIXED FIRING-TIMES

Let contestant A be armed with an automatic weapon which he fires continuously with a conds between rounds and with fixed-kill probability p_A. Contestant B, on the other hand, has

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a nonautomatic weapon which he fires periodically with fixed-kill probability p_B. B's firing time is a continuous random variable with probability density function $f_B(t)$.

Following the same analytical technique used earlier [1], we have immediately that the characteristic function of A's time to kill is

$$\Phi_A=\frac{p_A \exp(iau)}{1-q_A \exp(iau)}, \tag{1}$$

and similarly for B,

$$\Phi_B=\frac{p_B}{1-q_B\phi_B(u)} \tag{2}$$

where $\exp(iau)$ and $\phi_B(u)$ are the characteristic functions of A and B's firing times, respectively, and $q=1-p$.

As previously shown in [3],

$$P[A]=\frac{1}{2\pi i}\int_L \Phi_A(-u)\Phi_B(u)\frac{du}{u} \tag{3}$$

where $P[A]$ is the probability that A wins the duel, $\int_L$ represents a contour integral in the complex plane around the contour of Figure 1, where ρ is finite but less than the distance to the nearest singularity in the lower half-plane, and where $\int_L$ must vanish on C as $R\to\infty$. Substituting equations (1) and (2) into (3),

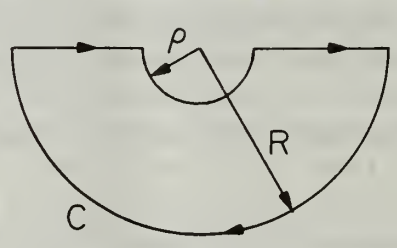

FIGURE 1. Path of integration in the complex plane.

$$P[A]=\frac{1}{2\pi i}\int_L \frac{p_A \exp(-iau)p_B\phi_B(u)}{[1-q_A \exp(-iau)][1-q_B\phi_B(u)]}\frac{du}{u}. \tag{4}$$

There is zero probability of a tie (because of B's continuous firing time); therefore, $P[B]=1-P[A]$, and the solution is complete.

Now, to illustrate the use of these expressions let

$$f_B(t)=r_B \exp(-r_B t) \tag{5}$$

where the parameter r_B is the reciprocal of the mean firing time and shall be called B's rate of fire. The characteristic function of (5) is

$$\phi_B(u)=\frac{r_B}{r_B-iu}. \tag{6}$$

Substituting equation (6) into (4),

$$P[A]=\frac{1}{2\pi i}\int_L \frac{p_A p_B r_B \exp(-iau)}{[1-q_A \exp(-iau)](p_B r_B-iu)}\frac{du}{u}. \tag{7}$$

'he integral $\to 0$ on C (see Figure 1) as $R\to\infty$ and it has one simple pole at $u=-ip_Br_B$. Using the ›sidue theorem we have

$$P[A]=\frac{p_A \exp(-ap_Br_B)}{1-q_A \exp(-ap_Br_B)}$$

)

$$=\frac{p_A \exp(-p_Br_B/r_A)}{1-q_A \exp(-p_Br_B/r_A)}$$

here we make the natural substitution

$$\frac{1}{a}=r_A,$$

nd where r_A is defined as A's rate of fire.

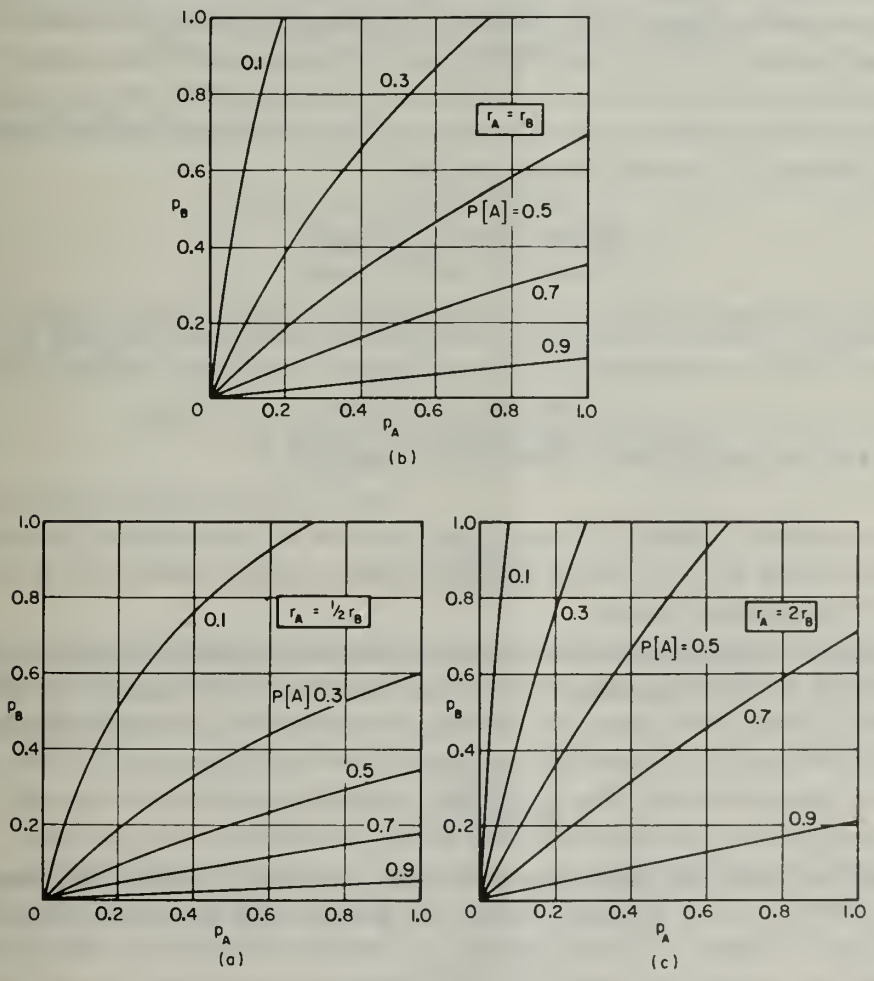

FIGURE 2. Fixed fire versus exponential random fire.

Equation (8) is plotted in Figure 2 as contour maps of $P[A]$ for three relative rates of fire. omparing Figure 2 with the corresponding Figure 2 in reference [3], where both sides have random ring times, we observe that all the contours for our case appear to be rotated clockwise somewhat ith respect to the corresponding contours in the earlier paper. This indicates that A does worse the case investigated here. The only difference in the two duels is that A now has a fixed rate fire which we take to be equal to his average rate of fire when his firing time is a random variable.

This hypothesis may be verified by investigating the assumption that

$$\frac{p_A}{\exp(p_B r_B/r_A)-q_A} < \frac{p_A r_A}{p_A r_A + p_B r_B}$$

where the right hand side is $P[A]$ for the duel in which both sides have negative exponential firing times [3]. This reduces to

$$1+\frac{p_B r_B}{r_A} < \exp(p_B r_B/r_A)$$

which is obviously true.

We may also compare A's performance in this case with the situation where both have fixed firing times by comparing Figure 2 with Figure 1 in reference [1]. Here the situation seems less clear. It appears that if $r_A < r_B$, A (fixed firing time) is better against random firing times than against fixed firing times, but when $r_A > r_B$ it may not necessarily be so. To test this we use the fundamental duel with fixed firing times from reference [1], and first let a be an integral multiple of b or $r_B = \alpha r_A$ where α is a positive integer, and test

$$\frac{p_A \exp(-p_B{}^{\alpha})}{1-q_A \exp(-p_B{}^{\alpha})} \geq \frac{p_A q_B{}^{\alpha}}{1-q_A q_B{}^{\alpha}},$$

which reduces to $\exp(-p_B) \geq 1-p_B$, which is true. For the case $r_A = \beta r_B$, where β is a positive integer, it is easy to show that sometimes A is better against a random firer and sometimes he is not.

THE DUEL WITH FIXED BURSTS RANDOMLY SPACED

Now, let us consider the possibly more realistic situation where contestants armed with automatic weapons fire fixed bursts of length N (i.e. N rounds, equally spaced, fired in a burst) and where the bursts are randomly spaced.

First, we observe a marksman firing at a target with a fixed hit probability, p, on each round fired. He fires bursts of N rounds, each round in the burst being fired at a random time after the preceding round and each burst begun at a random time from the termination of the preceding burst. Let $g(t)$ be the density function of times between rounds in a burst and $f(t)$ be the density function of times between bursts. The special case we are interested in is when $g(t)$ is a Delta Function (i.e. time between rounds is fixed), and we shall specialize to that case later.

The probability density function of time to a hit can be derived by noting that the probability of a hit in the interval $(t, t+dt)$ *and* that the hit is on the jth round of the kth burst is the probability of failing on all N rounds of each of the $(k-1)$ preceding bursts and on the preceding $j-1$ rounds of the kth burst times the probability of a hit times the probability that the hit occurs in the interval $(t, t+dt)$. This latter probability comes from summing k selections from $f(t)$ (time between bursts) and $(k-1)(N-1)+j-1$ selections from $g(t)$ (time between rounds within a burst). This accounts for the fact that the first round in a burst is fired immediately after the delay between bursts and only $N-1$ rounds in a burst are spaced according to $g(t)$. From the preceding we have $P[t<T<t+dt$ and a hit on j^{th} round of k^{th} burst$]=$

$$pq^{(k-1)N+j-1} \overset{k^*}{f}(t)^* g(t)^{[(k-1)(N-1)+j-1]^*} \tag{9}$$

where T is the random variable, time to a hit, * denotes a convolution, and k^* denotes k convolutions of a function with itself.

From (9) we may now compute the total probability of a hit in the interval $(t, t+dt)$ (omitting the factor dt on both sides) by

$$h(t)=p\sum_{k=1}^{\infty}\sum_{j=1}^{N} q^{(k-1)N+j-1} f^{k*}(t)*g(t)^{[(k-1)(N-1)+j-1]*} \tag{10}$$

where $h(t)$ is the density function of time to a hit.

If we let $\phi(u)$, $\psi(u)$, and $\Phi(u)$ be the characteristic functions of $f(t)$, $g(t)$, and $h(t)$, respectively, and if we take the Fourier transform of both sides of (10), then

$$\Phi(u)=p\sum_{k=1}^{\infty}\sum_{j=1}^{N} q^{(k-1)N+j-1}\phi^{k}(u)\psi(u)^{[(k-1)(N-1)+j-1]}. \tag{11}$$

Completing the double sum indicated in (11),

$$\Phi(u)=\frac{p\phi(u)[1-q^N\psi^{N}(u)]}{[1-q\psi(u)][1-q^N\phi(u)\psi^{N-1}(u)]}. \tag{12}$$

For the case where rounds in a burst are equally spaced, a time units apart, $\psi(u)=\exp(iau)$, which substituted into (12) is,

$$\Phi(u)=\frac{p\phi(u)[1-q^N\exp(iauN)]}{[1-q\exp(iau)][1-q^N\phi(u)\exp(ia(N-1)u)]}. \tag{13}$$

In passing, we note that if $N=1$ in (13), then

$$\Phi(u)=\frac{p\phi(u)}{1-q\phi(u)},$$

as it should [3], because this reduces our solution to the fundamental situation where there are no bursts. On the other hand, if $N\to\infty$ (which means the marksman fires one long burst after a random preparation time) then,

$$\Phi(u)=\frac{\phi p(u)}{1-q\exp(iau)},$$

which is awkward to deal with as it has an essential singularity in the lower half of the complex plane. And, further, if $a\to 0$ (no time between rounds in a burst),

$$\Phi(u)=\frac{\phi(u)(1-q^N)}{1-q^N\phi(u)},$$

which it should, since this is essentially the fundamental problem with p replaced by $1-q^N$ (N rounds all fired at once are like one round with this equivalent p).

Turning now to the duel, it is evident from (13) that if we attempt a burst-firing contestant versus a burst-firing opponent we shall face extremely difficult integration problems because of the essential singularities involved. However, if we pit a burstfirer versus a single-round-at-a-time firer, the problem is much simpler and we can still see the effects of burst firing. Consequently, we

let A fire randomly spaced, fixed-length bursts and let B fire single rounds randomly spaced. Thus, $\Phi_A(u)$ will be of the form of equation (13) with appropriate subscripts, and $\Phi_B(u)$ will be the same as equation (2) above. When these are substituted into equation (3),

$$P[A]=\frac{1}{2\pi i}\int_L \frac{p_A\phi_A(-u)[1-q_A{}^N \exp(-iauN)]p_B\phi_B(u)}{[1-q_A \exp(-iau)][1-q_A{}^N\phi_A(-u)\exp(-iau(N-1))](1-q_B\phi_B(u))}\frac{du}{u}. \tag{14}$$

Again, the probability of a draw is zero, and therefore $P[B]=1-P[A]$ and the solution is complete.

To illustrate this solution, let A's time between bursts and B's time between rounds be negative exponential, and $\phi_A(u)$ and $\phi_B(u)$ will have the form of equation (6) with parameters ρ_A and r_B respectively. It should be noted that ρ_A is A's rate of firing *bursts*, and $\frac{1}{a}$ is his rate of firing within bursts. Any attempt to define his overall rate of fire would appear to be pointless, as it would have to be some arbitrary combination of these parameters. When we make the indicated substitutions in equation (14),

$$P[A]=\frac{p_A\rho_A p_B r_B}{2\pi i}\int_L \frac{[1-q_A{}^N \exp(-iaNu)]}{[1-q_A{}^N \exp(-iau)][\rho_A+iu-q_A{}^N\rho_A \exp(-ia(N-1)u)](r_B-iu-q_Br_B)}\frac{du}{u}. \tag{15}$$

The integrand has one simple pole at $u=-i\,p_Br_B$ and, from the residue theorem we have,

$$P[A]=\frac{p_A\rho_A[1-q_A{}^N \exp(-Nap_Br_B)]}{[1-q_A \exp(-ap_Br_B)][\rho_A+p_Br_B-q_A{}^N\rho_A \exp(-(N-1)ap_Br_B)]}. \tag{16}$$

Although this is a rather complicated expression, it does throw some light on our topic. If $N=1$, it reduces to the solution for the fundamental duel (see [3]).

For $N\to\infty$,

$$P[A]=\frac{p_A\rho_A}{[1-q_A \exp(-ap_Br_B)](\rho_A+p_Br_B)}. \tag{17}$$

The only difference between this duel and the one in the first part of the paper is that A has an initial random delay for firing his first round instead of a fixed one. Comparing the two solutions and using subscripts 1 and 2 to distinguish between the first and second solutions respectively, let us see if $P[A]_2>P[A]_1$, or using equations (17) and (8),

$$\exp(ap_Br_B)>1+\frac{p_Br_B}{\rho_A} \tag{18}$$

which is certainly true if $\frac{1}{\rho_A}\leq a$. We must remember that $\frac{1}{\rho_A}$ is from (17) and is the average time between bursts. It is still true for even larger values of $\frac{1}{\rho_A}$, i.e. up to

$$\frac{\rho}{\rho_A}<\frac{1}{p_Br_B}^{[\exp(ap_Br_B)-1]} \quad \text{or} \tag{19}$$

$$\frac{1}{\rho_A}<a\left(1+\sum_{i=2}^{\infty}\left(\frac{ap_Br_B}{i!}\right)^{i-1}\right).$$

till for $N\to\infty$, if we let $\rho_A=r_A$ and compare our solution with the fundamental duel (with both ring-time distributions as negative exponential), we have $P[A]_f>P[A]_2$ as a hypothesis, where ubscript f refers to the fundamental duel. This reduces to

$$\exp(ap_Br_B)>1+\frac{p_Br_B}{r_A},$$

nd the situation is the same as that immediately preceding.

Continuing, if $a\to 0$, of course A will do better in burst firing than in single firing provided p_A the same in both cases. However, if burst firing degrades A's kill probability, then his degraded robability must be greater than $1-(q_A)^{1/N}$, where $q_A=1-p_A$ is his failure probability in the single-ring case. This follows easily from the known solutions. On the other hand, if $a\to\infty$,

20) $$P[A]=\frac{p_A\rho_A}{\rho_A+p_Br_B},$$

which means A only gets 1 round and, of course, his chances are severely degraded.

It now remains to examine equation (16) for the most interesting case, i.e., where N is moderate say 3) and all other factors are left open. Equation (16) is now rewritten, letting $N=3$, as

21) $$P[A]=\frac{\rho_A a p_A\{1-[q_A\exp(-ap_Br_B)]^3\}}{[1-q_A\exp(-ap_Br_B)]\{ap_Br_B+\rho_A a[1-q_A(q_A\exp(-ap_Br_B))^2]\}}.$$

rom this expression we see that we need consider only three dimensionless groups of parameters, amely, ap_Br_B, $\rho_A a$, and p_A. The physical meanings of the first two are; ratio of B's *hit* rate (p_Br_B) A's firing rate within bursts $\left(\frac{1}{a}\right)$, and A's rate between bursts (ρ_A) to A's firing rate within bursts $\left(\frac{1}{a}\right)$. Equation (21) is plotted in Figure 3 below for representative values of the parameters.

This figure illustrates in a quantitative way how variations in the parameters affect A's chances f winning. For example, looking at any one of the figures for ap_Br_B fixed and letting a be fixed, hen as $a\rho_A$ increases, we have ρ_A (A's burst rate) increasing, and we can see how $P[A]$ increases.)r, in going from Figure 3(a) to 3(b) to 3(c) for fixed a, p_Br_B is decreasing and $P[A]$ is increasing. 'o see the effect of changing only a, one must fix p_Br_B, ρ_A and p_A and move from figure to figure. 'or example, let p_Br_B and ρ_A both be one and $p_A=0.5$, then in Figure 3(a) $a=2$, $a\rho_A=2$, and $P[A]=$ .27. Moving to Figure 3(b), $a=1$, $a\rho_A=1$, and $P[A]=0.32$, and in Figure 3(c), $a=\frac{1}{2}$, $a\rho_A=\frac{1}{2}$, and $P[A]=0.35$.

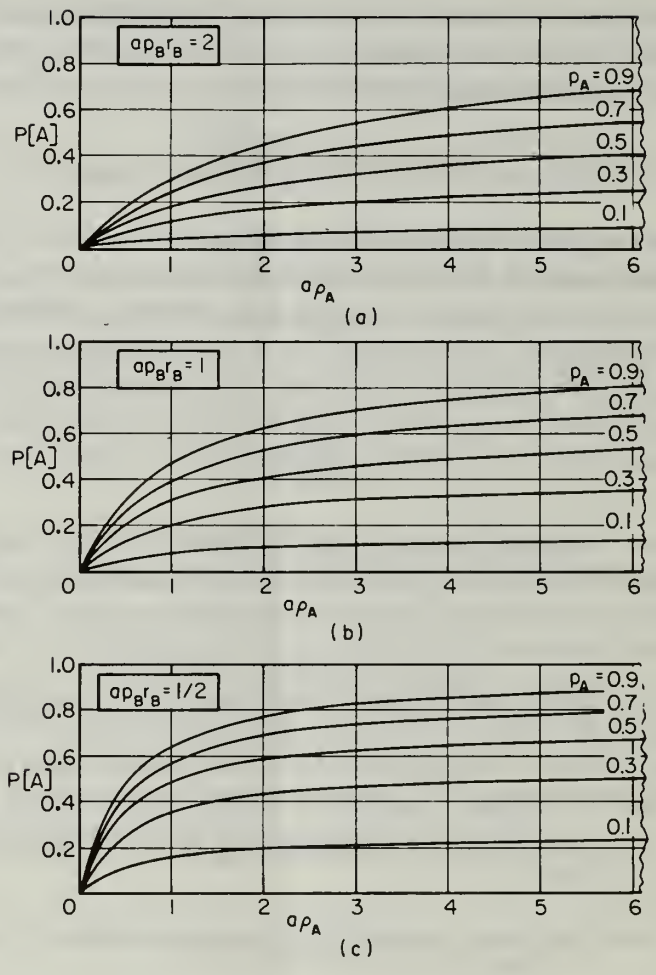

FIGURE 3. Exponential burst firer versus exponential random firer.

ONCLUSION

In the analysis above, we have compared burst firing (fixed times between rounds) with ran-om firing times under various conditions. In the single long burst situation where all other factors re the same, the burst firer (A) does worse against a random firer than if A were a random firer whose firing rate is the same. On the other hand, a single long-burst firer versus a single long-burst rer may or may not do better than a single long-burst firer versus a random firer depending on hat the values of the parameters are.

In the case of a contestant who fires fixed bursts, randomly spaced, against a random firer we ave seen what the effects of burst rate and rate of firing within bursts are.

Finally, we may mention that the foregoing shows once again that carelessly formulated nean-value models can give results which are grossly in error, because, if one considers fixed firing ates as a mean-value approximation of random firing rates, the results can be vastly different han the correct solutions using random variables give.

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A NOTE ON MINIMUM COST CONVEX FLOWS

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This note describes a slight extension of an algorithm developed by T. C. Hu [1], for finding minimum-cost network flows with nonnegative convex costs. Here we show that the assumption that the costs are nonnegative can be relaxed, and with minor modifications the algorithm presented in [1] remains valid.

Consider a network for nodes N_k, $k=1, 2, \ldots, n$ and arcs A_{ij} joining nodes N_i and N_j. Associated with each arc A_{ij} is an arbitrary convex cost function $C_{ij}(x_{ij})$. Associated with each node N_k is an integer T_k representing the net demand at node k. ($T_k<0$ indicates a surplus at N_k.) We will assume the network is balanced; i.e., $\sum_{k=1}^{n} T_k=0$. The problem is to find a set of arc flows x_{ij} which minimizes

$$\sum_{(i,j)} C_{ij}(x_{ij}) \tag{1}$$

while satisfying

$$\sum x_{ik}-\sum_{j} x_{kj}=T_k \qquad k=1, 2, \ldots, n \tag{2}$$

and

$$x_{ij}\geq 0 \qquad \text{for all arcs } A_{ij}. \tag{3}$$

Problems fitting the model (1), (2), and (3) with negative costs do arise in the following context. Suppose that the first b_{ij} units of flow sent along A_{ij} generate a unit revenue of r_{ij} whereas additional units generate a unit cost h_{ij}. Then, the cost functions are

$$C_{ij}(x_{ij})=\begin{cases} -r_{ij}x_{ij} & 0\leq x_{ij}\leq b_{ij} \\ -r_{ij}b_{ij}+h_{ij}(x_{ij}-b_{ij}) & x_{ij}>b_{ij} \end{cases}$$

These functions are convex and do have negative values. A problem of this type cannot be solved by the algorithm of [1] but can be solved using the modifications presented in this note.

The assumption that $C_{ij}(x_{ij})\geq 0$ simplifies the problem since an optimal solution can always be found in which $x_{ij}=0$ or $x_{ji}=0$, for all i and j. This is not always true if we allow some or all of the cost functions to take on negative values. In order for the solution to be bounded we must stipulate that $C_{ij}(x_{ij})$ is bounded from below.

The algorithm of [1] finds a feasible set of flows x_{ij} and then tests for optimality by computing the incremental cost of sending a unit of flow from N_i to N_j for all i and j. The incremental cost may involve the cost of cancellation of existing flow from N_j to N_i. The optimality test involves a search for negative cycles based on these incremental costs.

The incremental costs in [1] are defined in terms of an "up-cost" u_{ij} and a "down-cost" d_{ij}, defined as

$$u_{ij}(x_{ij}) = C_{ij}(x_{ij}+1) - C_{ij}(x_{ij}), \text{ and} \tag{4}$$

$$d_{ij}(x_{ji}) = -C_{ji}(x_{ji}) + C_{ji}(x_{ji}-1). \tag{5}$$

When negative values are allowed, the incremental cost of sending a unit of flow from N_i to N_j may be either the cost of increasing x_{ij} or the cost of decreasing x_{ji}, whichever is smallest. The incremental cost can be defined in terms of u_{ij} and d_{ij} as

$$C^*_{ij}(x_{ij}) = \min\,[u_{ij}(x_{ij}),\ d_{ij}(x_{ji})]. \tag{6}$$

The modified algorithm can be stated as follows. Find a set of feasible flows x_{ij} satisfying (2) and (3). Compute C^*_{ij} using (4), (5), and (6) for all arcs A_{ij}. Search for negative cycles based on the C^*_{ij} costs. If none exist, stop; otherwise generate an improved set of flows and repeat the process.

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